NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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INTRODUCTION

In the sequel, K is a non-archimedean valued field, complete, with residue class field k. Our aim is to present reasonable definitions for a function $f : X \to K$ to be "monotone". (X is a subset of K). Since K admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \to \mathbb{R}$, equivalent to "f is monotone", and such that these statements have translations in K that make sense. This way we obtain several definitions of "$f : X \to K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of K", "the sign of a nonzero element of K".

In Section 2 we define several notions of monotony. E.g., $f \in M_b(X)$ if x between y and z implies f(x) between f(y) and f(z) and $f \in M_s(X)$ if f(x) between f(y) and f(z) implies x between y and z. Also monotone functions of type σ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M_b(X)$ (or $f \in M_s(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-information on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of K and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: $f' > 0 \iff f$ increasing.

The notion of pseudo-ordening ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

Notations. Let p be a prime. By $\mathbb{F}_p$ we mean the field of p elements. By $\mathbb{Q}_p$ the non-archimedean valued field of the p-adic numbers. For a field L we denote its characteristic by $\chi(L)$. Let E be a vector space over K and $S \subset E$. By $[\![S]\!]$ we denote the smallest K-linear subspace of E that contains S.

1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let $x,y \in K$. Then the smallest ball in K containing x and y is denoted by $[x,y]$. A subset C of K is called convex if $x,y \in C$ implies $[x,y] \subset C$.

Sometimes we use a more geometric terminology. Instead of $z \in [x,y]$ we will say that z is between x and y and instead of $z \notin [x,y]$ we use the expression: x and y are at the same side of z. Notice that $[x,y] = [y,x]$ for all $x,y \in K$ and that $z \in [x,y] \Longleftrightarrow |z-x| \le |x-y| \Longleftrightarrow |z-y| \le |x-y| \Longleftrightarrow z = \lambda x+(1-\lambda)y$ for some $\lambda \in K$, $|\lambda| \le 1$. If $x \neq y$ then the λ in this last expression is unique (viz. $\lambda = \frac{z-y}{x-y}$).

Examples of convex sets are: the empty set, singletons, balls, K. It is an easy exercise to show that these are the only convex subsets of K. So formally we may write each convex subset of K as

$$\{x \in K : |x-a| < r\} \qquad (a \in K,\ 0 \le r \le \infty)$$

or as

$$\{x \in K : |x-a| \le r\} \qquad (a \in K,\ 0 \le r \le \infty)$$

Notice that the only unbounded convex subset of K is K itself.

Sometimes we need the notion of convexity with respect to a subset X of K. A subset $C \subset X$ is called convex in X if $x,y \in C$ implies $[x,y] \cap X \subset C$ or, equivalently, if C is the intersection of X with a convex subset of K.

Let $x,y,z \in K$. By the strong triangle inequality we have that the "triangle" x,y,z is isosceles, say $|x-y| = |y-z|$. Then $|x-z| \le |x-y|$, so z is between x and y and x is between y and z. If also $|x-y| = |x-z|$

then y is between x and z. Otherwise, x and z are at the same side of y.

The relation $\sim$ defined on $K^* := K\setminus\{0\}$ by

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x,y \in K^*)$$

is an equivalence relation. We have $x \sim y$ iff $0 \notin [x,y]$ i.e. iff $|x-y| < |x|$ $(= |y|)$ i.e. iff $|xy^{-1}-1| < 1$. Define

$$K^+ := \{x \in K : |1-x| < 1\}$$

Then K^+ is a multiplicative subgroup of K^*; $K^+ = \{x \in K^* : x \sim 1\}$ and is called the set of the <u>positive</u> elements of K. The relation $\sim$ is also induced by the canonical group homomorphism

$$\pi : K^* \to K^*/K^+.$$

Thus, $x \sim y$ if and only if $\pi(x) = \pi(y)$ $(x,y \in K^*)$. Therefore it is natural to view the group $\Sigma := K^*/K^+$ as being the <u>group of signs</u> of elements of K^*, and we call $\pi(x)$ the <u>sign of the element</u> $x \in K^*$. If $x \in K^*$ then $\pi(x) = \{y : |y-x| < |x|\} = xK^+$. For $x \in K^*$, $\alpha \in \Sigma$ we sometimes write $x\alpha$ to indicate the element $\pi(x).\alpha$ of Σ. In particular, for $\alpha \in \Sigma$ the <u>opposite sign</u> of α, $-\alpha$, is defined as $(-1)\alpha$. Then $-\alpha = \{-x : x \in \alpha\}$. (Notice that in case $\chi(K) = 2$ we have $\alpha = -\alpha$.)

Let $\alpha \in \Sigma$. Then for $x,y \in \alpha$ we have $|x| = |y|$ so we can define the <u>absolute value of</u> α, $|\alpha|$ as follows

$$|\alpha| := |x| \qquad (x \in \pi^{-1}(\alpha)).$$

In the sequel we also need addition between elements of Σ. Let us first observe that for any $\alpha,\beta \in \Sigma$ the sum

$$\alpha+\beta := \{x+y : x \in \alpha,\ y \in \beta\}$$

is always a ball with radius $\max(|\alpha|^-,|\beta|^-)$. (I.e., of the form

$\{x : |x-b| < \max(|\alpha|,|\beta|)\})$. Now $\alpha+\beta$ contains 0 if and only if $\alpha = -\beta$. Otherwise $\alpha+\beta$ is again an element of Σ. (Proof: Let $a \in \alpha$, $b \in \beta$. Then $|a+b| = \max(|a|,|b|)$. If also $x \in \alpha$, $y \in \beta$ then $|x+y-(a+b)| \leq \max(|x-a|,|y-b|) < \max(|a|,|b|) = |a+b|$. Thus $\alpha+\beta$ contains the ball with center $a+b$ and radius $\max(|\alpha|^-,|\beta|^-)$, so $\alpha+\beta$ is equal to this ball.)

Let us define

$$\alpha \oplus \beta := \alpha+\beta = \{x+y : x \in \alpha, y \in \beta\} \qquad (\alpha,\beta \in \Sigma, \alpha \neq -\beta).$$

We have

THEOREM 1.2 *Let* Σ, $|\ | : \Sigma \to \mathbb{R}$, $\oplus : \Sigma \times \Sigma \setminus \{(\alpha,-\alpha) : \alpha \in \Sigma\} \to \Sigma$ *be as above. Let* $\alpha,\beta,\gamma \in \Sigma$. *Then*

(i) $|\alpha\beta| = |\alpha|\ |\beta|$, $|\alpha^{-1}| = |\alpha|^{-1}$.

(ii) *If* $\alpha \oplus \beta$ *is defined then so is* $\beta \oplus \alpha$ *and* $\alpha \oplus \beta = \beta \oplus \alpha$.

(iii) *If* $(\alpha \oplus \beta) \oplus \gamma$ *and* $\alpha \oplus (\beta \oplus \gamma)$ *are defined then* $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$.

(iv) *If* $\alpha \oplus \beta$ *or* $\gamma\alpha \oplus \gamma\beta$ *is defined then so is the other and* $\gamma(\alpha \oplus \beta) = \gamma\alpha \oplus \gamma\beta$.

(v) *If* $\alpha \oplus \beta$ *is defined then* $|\alpha \oplus \beta| = \max(|\alpha|,|\beta|)$. *Conversely if* $|s| = \max(|\alpha|,|\beta|)$ *for some* $s \in \alpha+\beta$ *then* $\alpha \oplus \beta$ *is defined.*

(vi) $|\alpha| < |\beta|$ *if and only if* $\alpha \oplus \beta = \beta$.

(vii) *Let* $n \in \{1,2,\ldots,\chi(k)-1\}$ *if* $\chi(k) \neq 0$, *let* $n \in \mathbb{N}$ *otherwise. Then we define* $\oplus_n \alpha$ *inductively as follows.* $\oplus_1 \alpha : \alpha$, $\oplus_k \alpha := \oplus_{k-1} \alpha \oplus \alpha$ $(k \leq n)$. *Then*

$$\oplus_n \alpha = n\alpha.$$

Proof. (i), (ii) are clear. (iii) is almost trivial: if $x \in \alpha$, $y \in \beta$, $z \in \gamma$ then $x+y+z \in \alpha+\beta+\gamma$ and the latter set can be regarded as

$(\alpha \oplus \beta) \oplus \gamma$ or as $\alpha \oplus (\beta \oplus \gamma)$. (It is worth noticing that $(\alpha \oplus \beta) \oplus \gamma$ may be defined whereas $\alpha \oplus (\beta \oplus \gamma)$ is not. Choose $\beta = -\gamma$ and $|\alpha| > |\beta|$. Then $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus \gamma = \alpha$, $\beta \oplus \gamma$ is not defined.) (iv) is clear. If $\alpha \oplus \beta$ is defined then for $x \in \alpha$, $y \in \beta$ we have $|x+y| \geq \max(|x|,|y|)$ whence $|x+y| = \max(|x|,|y|)$. So $|\alpha \oplus \beta| = \max(|\alpha|,|\beta|)$. Conversely, if $\alpha \oplus \beta$ is not defined, then (we saw earlier that) $\alpha+\beta$ is a ball with center zero and radius $\max(|\alpha|^-, |\beta|^-)$. Thus we have (v). We prove (vi). If $|\alpha| < |\beta|$ then $\alpha+\beta = \beta$ so $\alpha \oplus \beta = \beta$. Conversely, if $\alpha \oplus \beta = \beta$ then choose $a \in \alpha$, $b \in \beta$. Then $a+b \in \beta$ hence $a+b \sim b$ i.e., $ab^{-1}+1 \in K^+$ implying $|ab^{-1}| < 1$ or $|a| < |b|$. Hence $|\alpha| < |\beta|$. (Note: from (vi) it follows that $\alpha \oplus \beta = \alpha' \oplus \beta$ does not imply $\alpha = \alpha'$.) To prove (vii) let $a \in \alpha$ and observe that for any $k \leq n$, if $\bigoplus_{k-1} \alpha$ is defined, $(k-1)a \in \bigoplus_{k-1} \alpha$. Hence $|(k-1)a+a| = |ka| = |a| = |\alpha|$, so $\bigoplus_{k-1} \alpha+\alpha$ does not contain 0, hence $\bigoplus_{k-1} \alpha \oplus \alpha$ is defined.
Now $n\alpha$ is by definition $\pi(n)\alpha$. So $na \in n\alpha$ and $na \in \bigoplus_{n} \alpha$. Since both $n\alpha$ and $\bigoplus_{n} \alpha$ are signs they must coincide.

We now define relations that resemble "ordering".

DEFINITION 1.3 <u>Let</u> $\alpha \in \Sigma$ <u>and</u> $x,y \in K$. <u>Then we say that</u> x <u>is greater than</u> y <u>in the sense of</u> α, <u>notation</u> $x >_\alpha y$, <u>if</u> $x-y \in \alpha$.

We have the following rules

THEOREM 1.4 (i) <u>If</u> $x,y \in K$, $x \neq y$ <u>then there is exactly one</u> $\alpha \in \Sigma$ <u>for which</u> $x >_\alpha y$.

(ii) $x >_\alpha x$ <u>for no</u> α.

(iii) <u>If</u> $x >_\alpha y$ <u>then for all</u> $s \in K$: $x+s >_\alpha y+s$ $(x,y \in K,\ \alpha \in \Sigma)$

(iv) <u>If</u> $x >_\alpha y$ <u>and</u> $s >_\beta 0$ <u>then</u> $xs >_{\alpha\beta} ys$ $(x,y,s \in K,\ \alpha,\beta \in \Sigma)$

(*In particular* $x >_\alpha y$ *implies* $-x >_{-\alpha} -y$).

(v) *If* $x >_\alpha y$, $y >_\beta z$ *and if* $\alpha \oplus \beta$ *is defined then* $x >_{\alpha\oplus\beta} z$.

Proof. Easy.

The group $\Sigma_1 := \{\alpha \in \Sigma : |\alpha| = 1\}$ is a subgroup of Σ, isomorphic to multiplicative group k^*. If K has discrete valuation and if $s \in K$ and $|s|$ is the largest value that is smaller than 1, then for each $\alpha \in \Sigma$ there is $x \in \mathbb{Z}$ such that $\alpha = s^n\alpha_1$ where $\alpha_1 \in \Sigma_1$. It follows easily that the map $(n,\alpha) \mapsto s^n\alpha$ $(n \in \mathbb{Z},\ \alpha \in \Sigma_1)$ is an isomorphism of $\mathbb{Z} \times \Sigma_1$ onto Σ. Thus, *in case* K *has discrete valuation*, Σ *is isomorphic to* $\mathbb{Z} \times \Sigma_1$, or, *for that matter, to* $|K^*| \times k^*$.

If K is a local field we can even define a group embedding $\rho : \Sigma \to K^*$ such that $\pi\rho$ is the identity. (Thus, we can pick an element in every α $(\alpha \in \Sigma)$ such that the resulting set is a subgroup of K^*). Let $s \in K$, $|s| < 1$ such that $|s|$ generates the value group and let q be the cardinality of k. Let $x \in K^*$. Then there is a unique $n \in \mathbb{Z}$ such that $x = s^n x_1$ where $|x_1| = 1$.

Define
$$\nu(x) = s^n \lim_{n\to\infty} x_1^{q^n}$$

It is easy to verify that ν is a homomorphism of K^* into K^*, that $\pi(\nu(x)) = \pi(x)$ for all $x \in K^*$ and that $\nu(x) = 1$ if and only if $x \in K^+$. Therefore the map ρ making the diagram

$$\begin{tikzcd} K^* \arrow[rr, dashed, "\nu"] \arrow[dr, "\pi"'] & & K^* \\ & \Sigma \arrow[ur, dashed, "\rho"'] & \end{tikzcd}$$

commute solves the problem.

EXAMPLE 1.5 *The signs of* $\mathbb{Q}_p$. Let θ be a primitive $(p-1)^{\text{th}}$ root of unity. Then $\{\Theta^i p^n : i \in \{0,1,\dots,p-2\},\ n \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Q}_p^*$ isomorphic to Σ. If

$$x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0,1,\Theta,\dots,\Theta^{p-2}\},\ a_n \neq 0)$$

is an element of $\mathbb{Q}_p$, its sign, interpreted as an element of $\mathbb{Q}_p$ is

$$\pi(x) = a_n p^n.$$

2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function $f : [0,1] \to \mathbb{R}$ the following statements are equivalent.

(α) f is monotone (i.e., either $x > y$ implies $f(x) \geq f(y)$ for all x,y or $x > y$ implies $f(x) \leq f(y)$ for all x,y).

(β) If x is between y and z then $f(x)$ is between $f(y)$ and $f(z)$ ($x,y,z \in [0,1]$)

(γ) If $C \subseteq \mathbb{R}$ is convex then $f^{-1}(C)$ is convex.

Thus we define

DEFINITION 2.1 _Let_ $X \subset K$. $f : X \to K$. _We say that_ $f \in M_b(X)$ _if for all_ $x,y,z \in X$, x _between_ y _and_ z _implies_ $f(x)$ _is between_ $f(y)$ _and_ $f(z)$. _In other words,_ $f \in M_b(X)$ _if and only if for all_ x,y,z

$$|x-y| \leq |y-z| \to |f(x)-f(y)| \leq |f(y)-f(z)|.$$

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

THEOREM 2.2 _Let_ $X \subset K$ _and let_ $f : X \to K$. _Then the following statements are equivalent_

(α) $f \in M_b(X)$.

(β) _For each convex_ $C \subset K$, $f^{-1}(C)$ _is convex in_ X.

(γ) _For all_ $x,y,z \in X$: $|x-y| = |x-z| \to |f(x)-f(y)| = |f(x)-f(z)|$.

(δ) _For all_ $x,y,z \in X$: $|f(x)-f(y)| > |f(x)-f(z)| \to |x-y| > |x-z|$.

(ε) _For all_ $x,y,z \in X$: $|f(x)-f(y)| \neq |f(x)-f(z)| \to |x-y| \neq |x-z|$.

Proof. $(\alpha) \to (\beta)$. Let $x,y \in f^{-1}(C)$ and let $z \in [x,y] \cap X$. Then $|z-x| \le |x-y|$, so $|f(z)-f(x)| \le |f(x)-f(y)|$ i.e., $f(z) \in [f(x),f(y)] \subset C$. Hence $z \in f^{-1}(C)$.

$(\beta) \to (\alpha)$. Let $x,y,z \in X$ and $|x-y| \le |x-z|$. The set $[f(x),f(z)]$ is convex, hence $f^{-1}([f(x),f(z)])$ is convex in X and contains x and z, so it must contain y. Thus $f(y) \in [f(x),f(z)]$.

Clearly, $(\alpha) \leftrightarrow (\delta)$ and $(\gamma) \leftrightarrow (\varepsilon)$. We prove $(\alpha) \leftrightarrow (\gamma)$. Now $(\alpha) \to (\gamma)$ is clear by symmetry. Suppose (γ) and let $|x-y| \le |x-z|$. It suffices to consider the case $|x-y| < |x-z|$. Then $|y-z| = |x-z|$, so by (γ) we have $|f(y)-f(z)| = |f(x)-f(z)|$. Then $|f(x)-f(y)| \le \max(|f(x)-f(z)|,|f(z)-f(y)|) = |f(x)-f(z)|$.

THEOREM 2.3 _Let_ $X \subset K$. _Then_

(i) _For each_ $a,b \in K$ _the map_ $x \mapsto ax+b$ _is in_ $M_b(X)$.

(ii) _If_ $f \in M_b(X)$, $\lambda \in K$ _then_ $\lambda f \in M_b(X)$.

(iii) $M_b(X)$ _is closed under pointwise limits_.

(iv) _If_ $f \in M_b(X)$ _and_ $g : f(X) \to K$ _is in_ $M_b(f(X))$, _then_ $g \circ f \in M_b(X)$.

(v) _If_ $f \in M_b(X)$ _and_ $f(a) = f(b)$ _for some_ $a,b \in X$, _then_ f _is constant on_ $[a,b] \cap X$.

Proof. Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of M_b-functions. For more, see the sequel.

(1) The constant functions.

(2) Isometries (e.g., exp.).

(3) Choose in every $\alpha \in \Sigma$ an element x_α. Define $\phi : K \to K$ as follows

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_\alpha & \text{if } x \in \alpha \qquad (\alpha \in \Sigma) \end{cases}$$

(Essentially, $\phi|K^*$ is the sign function π of section 1).

We prove that $\phi \in M_b(K)$. Since ϕ is continuous it suffices to check that $\phi|K^*$ is in $M_b(K^*)$. Now for all $x,y \in K^*$ we have $\phi(x)-\phi(y) = 0$ if $|xy^{-1}-1| < 1$ and $|\phi(x)-\phi(y)| = |x-y|$ if $|x-y| = \max(|x|,|y|)$. Now take $x,y,z \in K^*$ such that $|x-y| \le |x-z|$. If $\phi(x) = \phi(z)$ then $|1-x^{-1}y| \le |1-x^{-1}z| < 1$ so $\phi(x) = \phi(y)$.

If $\phi(x) \neq \phi(z)$ then $|\phi(x)-\phi(y)| \le |x-y| \le |x-z| = |\phi(x)-\phi(z)|$.

(4) Let $r > 0$ and choose in every ball B of radius r a center x_B. The function ψ defined via

$$\psi(x) = x_B \qquad (x \in B)$$

is in $M_b(K)$. The proof is easy.

(5) (A nowhere continuous M_b-function). Let K be a field such that $\#K = \#k$ (e.g., a discretely valued field where $\#k$ has the power of the continuum). Let $\sigma : K \to k$ be a bijection and let $\tau : k \to K$ such that $|\tau x-\tau y| = 1$ whenever $x \neq y$. Then $f := \tau \circ \sigma$ satisfies: $|f(x)-f(y)| = 1$ $(x,y \in K,\ x \neq y)$.

Clearly f is everywhere discontinuous, $f \in M_b(K)$.

(6) Let $X \subset K$. We can strengthen the definition of an M_b-function into

$$\text{if } |x-y| \le |z-t| \text{ then } |f(x)-f(y)| \le |f(z)-f(t)| \qquad (x,y,z,t \in X)$$

(some "uniform" M_b-condition) and we obtain a space, called $M_{ub}(X)$. Clearly, the examples mentioned in (1),(2),(4),(5) are in $M_{ub}(K)$, whereas the example in (3) is not. (Choose $x,y \in K$ with $|x| > 1$, $|x-y| = 1$. Then $|1-0| \le |x-y|$, but $1 = |\phi(1)-\phi(0)| > |\phi(x)-\phi(y)| = 0$.) Notice that ϕ is locally constant on K^*, but not on K.

(7) The discontinuous function f of (5) has the property that $f(K)$ consists only of isolated points. This is not accidental. If $f \in M_b(K)$

if $f(a)$ is a non-isolated point of $f(K)$ and B is a ball containing $f(a)$ then $f^{-1}(B)$ is convex (2.2 (d)) and contains at least two points, so $f^{-1}(B)$ is an open set. Thus, f is continuous at a. It follows that the image of an everywhere discontinuous M_b-function consists only of isolated points.

(8) For each n, let $\sigma_n : K \to K$ be the example of 2.4, (4) above where $r = \frac{1}{n}$. Then σ_n is locally constant. For each $f \in M_b(X)$ we have $\sigma_n \circ f \in M_b(X)$ and $\lim \sigma_n \circ f = f$ uniformly. Hence, if f is continuous then it can uniformly be approximated by locally constant M_b-functions.

A monotone function $f : [0,1] \to \mathbb{R}$ maps convex sets into sets that are relatively convex in $f([0,1])$. In our situation we do not have such a property for M_b-functions. See 2.10 but also 5.2. If $f : [0,1] \to \mathbb{R}$ maps convex sets into (relatively) convex sets then f need not be monotone: any Darboux continuous function has the above property. In fact a function $f : [0,1] \to \mathbb{R}$ is Darboux continuous if and only if f maps convex sets into convex subsets of $\mathbb{R}$. We define

DEFINITION 2.5 <u>Let</u> X <u>be a subset of</u> K, <u>and let</u> $f : X \to K$. <u>Then</u> f <u>is called weakly Darboux continuous if for every relatively convex set</u> $C \subset X$ <u>the set</u> $f(C)$ <u>is convex in</u> $f(X)$. f <u>is called Darboux continuous if for every relatively convex set</u> $C \subset X$ <u>the set</u> $f(C)$ <u>is convex (in</u> K).

We have the following obvious remarks.

1) $f : X \to K$ is Darboux continuous if and only if f is weakly Darboux continuous and $f(X)$ is convex in K.

2) A Darboux continuous function need not be continuous. In fact we can construct an $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that for every open ball $B \subset \mathbb{Z}_p$, $f(B) = \mathbb{Z}_p$. Let $A \subset \mathbb{Z}_p$ be defined as follows. $x = \Sigma a_n p^n$ ($a_n \in \{0,1,\dots,p-1\}$) is in A if $a_{2n} = a_{2n+2} = \dots = 0$ for some n. Define $f : \mathbb{Z}_p \to \mathbb{Q}_p$ via

$$f(x) = \begin{cases} a_{2N+1}+a_{2N+3}p+a_{2N+5}p^2+\dots & \text{if } x \in A \text{ and } N = \min\{n : a_{2n} = a_{2n+2} = \dots = 0\} \\ 0 \quad \text{if } x \notin A \end{cases}$$

Then f maps every non empty open set onto $\mathbb{Z}_p$ (so f is Darboux continuous) but f is nowhere continuous.

(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on $\mathbb{Z}_p$ is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous. To see this, observe that all open compact subsets of $\mathbb{Z}_p$ are homeomorphic, so we can make a homeomorphism of $\mathbb{Z}_p$ onto $\mathbb{Z}_p$ sending $\{x : |x| < 1\}$ into $\{x : |x| = 1\}$ and $\{x : |x| = 1\}$ into $\{x : |x| < 1\}$. If $p \neq 2$ this map is not weakly Darboux continuous.

4) A Darboux continuous M_b-function is continuous. (Proof. f(X) is convex, so either f is constant or f(X) has no isolated points. By the remark made in 2.4,(7), f is continuous.)

Next, we consider translations of "strict monotony". For an $f : [0,1] \to \mathbb{R}$ the following conditions are equivalent.

(α) f is strictly monotone (i.e., injective and monotone).

(β) f is injective. For a convex set $C \subset [0,1]$, f(C) is convex in f([0,1]).

(γ) For all $x,y,z \in [0,1]$: if $f(x)$ is between $f(y)$ and $f(z)$ then x is between y and z.

(δ) For all $x,y,z \in [0,1]$: $f(x)$ is between $f(z)$ if and only if x is between y and z.

Translating (α) - (δ) into the non-archimedean situation we arrive at the following conditions. Let $X \subset K$ and $f : X \to K$

(α') $f \in M_b(X)$ and f is injective.

(β') f is weakly Darboux continuous and injective.

(γ') for all $x,y,z \in X$, $|x-y| < |x-z|$ implies $|f(x)-f(y)| < |f(x)-f(z)|$.

(δ') $f \in M_b(X)$ and f satisfies (γ').

It will turn out that the conditions (α') - (γ') although not equivalent are closely related. We start with (γ'):

DEFINITION 2.6 <u>Let</u> $X \subset K$, $f : X \to K$. <u>We say that</u> $f \in M_s(X)$ <u>if for all</u> x,y,z, $f(x) \in [f(y),f(z)]$ <u>implies</u> $x \in [y,z]$.

THEOREM 2.8 <u>Let</u> $X \subset K$, $f : X \to K$. <u>Then the following statements are equivalent</u>:

(α) $f \in M_s(X)$.

(β) f <u>is injective and weakly Darboux continuous</u>.

(γ) f <u>is injective and</u> $f^{-1} \in M_b(f(X))$.

(δ) <u>For all</u> $x,y,z \in X$ $|f(x)-f(y)| = |f(x)-f(z)| \to |x-y| = |x-z|$.

(ε) <u>For all</u> $x,y,z \in X$ $|x-y| < |x-z| \to |f(x)-f(y)| < |f(x)-f(z)|$.

(ζ) <u>For all</u> $x,y,z \in X$ $|x-y| \neq |x-z| \to |f(x)-f(y)| \neq |f(x)-f(z)|$.

Proof. The implications $(\alpha) \to (\varepsilon) \to (\zeta) \to (\delta)$ are clear from the definitions.

$(\delta) \to (\gamma)$: injectivity follows from $|f(x)-f(x)| = |f(x)-f(y)| \to |x-x| = |x-y|$. Use 2.2.$(\gamma)$.

$(\gamma) \to (\beta)$: Let $g : f(X) \to X$ be the inverse of f. Let $C \subset X$ be convex in X. Then since $g \in M_b$, $g^{-1}(C)$ is convex in $f(X)$. But $g^{-1}(C) = f(C)$.

Finally, we prove $(\beta) \to (\alpha)$. Let $f(x) \in [f(y),f(z)]$. By (β) the set $f([y,z] \cap X)$ is convex in $f(X)$ and it contains $f(y),f(z)$, hence $f(x) \in [f(y),f(z)] \cap X \subset f([y,z] \cap X)$. Since f is injective, $x \in [y,z] \cap X$ and we are done.

We also have (compare 2.3)

THEOREM 2.9 Let $X \subset K$. Then

(i) For $a,b \in K$, $a \neq 0$ the map $x \mapsto ax+b$ is in $M_s(X)$.

(ii) If $f \in M_s(X)$, $\lambda \in K$, $\lambda \neq 0$ then $\lambda f \in M_s(X)$.

(iii) If $f_1, f_2, \ldots \in M_s(X)$, $\lim f_n = f$ pointwise, f injective then $f \in M_s(X)$.

(iv) If $f \in M_s(X)$, $g \in M_s(f(X))$ then $g \circ f \in M_s(X)$.

Proof. Obvious verifications.

Returning to our conditions (α') - (δ') we see that (β') is equivalent to (γ'), that (α') means $f^{-1} \in M_s(f(X))$ and that (δ') menas $f \in M_b(X) \cap M_s(X)$.

Our f of example 2.4 (5) is in M_b, injective but not in M_s. Its inverse yields an example of an M_s-function that is not in M_b. Thus, in general, we have neither one of the implications $(\alpha') \to (\gamma')$, $(\gamma') \to (\alpha')$, $(\beta') \to (\delta')$, $(\alpha') \to (\delta')$. But our counterexample is rather weird (f is nowhere continuous and the domain of f^{-1} is discrete). We can do better.

EXAMPLE 2.10 <u>Let K have discrete valuation and let k be infinite. Then there exists a homeomorphism of the unit ball of K that is in M_b but not in M_s. (The inverse map is in M_s but not in M_b).</u>

<u>Proof</u>. Set $X = \{\alpha \in K : |\alpha| \le 1\}$ and let R be a full set of representatives of the equivalence relation $x \sim y$ iff $|x-y| < 1$ in X. Then R is infinite. Let $\pi \in K$ be such that $|\pi|$ is the largest value that is smaller than 1. The map

$$(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \qquad (a_i \in R \text{ for each } i)$$

is a bijection of $R^{\mathbb{N}}$ onto X. We may suppose that $0 \in R$.

Since R is infinite we can define injections

$$\tau_1 : R \setminus \{0\} \to R$$

$$\tau_2 : R \to R$$

such that $\operatorname{im} \tau_1 \cap \operatorname{im} \tau_2 = \emptyset$, $\operatorname{im} \tau_1 \cup \operatorname{im} \tau_2 = R$.

For $x = \sum_{n=0}^{\infty} a_n \pi^n \in X$ ($a_n \in R$ for each n) set

$$f(x) := \begin{cases} \tau_1(a_0) + a_1\pi + \ldots = x - a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\ \tau_2(a_1) + a_2\pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0 \end{cases}$$

A simple inspection of the definition shows that f is a bijection of X onto X. If $a, b \in R$, $a \neq b$ then $|a\pi - b\pi| < |b\pi - a|$, whereas $|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1$ and $|f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1$, so $f \notin M_s(X)$. Finally, let $x, y, z \in X$ and $|x-y| \le |x-z|$. We prove that $|f(x)-f(y)| \le |f(x)-f(z)|$. If $|f(x)-f(z)| = 1$ there is nothing to prove, so suppose $|f(x)-f(z)| < 1$. Set $x = \Sigma a_n \pi^n$, $y = \Sigma b_n \pi^n$, $z = \Sigma c_n \pi^n$.

If $a_0 = 0$ then also $c_0 = 0$ and $\tau_2(a_1) = \tau_2(c_1)$ so $a_1 = c_1$, hence $|x-z| \le |\pi|^2$. Since $|x-y| \le |x-z|$ we have also $b_0 = 0$, $b_1 = a_1$. So, $f(x)-f(y) = \frac{x-y}{\pi}$, $f(x)-f(z) = \frac{y-z}{\pi}$ whence $|f(x)-f(y)| \le |f(x)-f(z)|$.

If $a_0 \ne 0$ then $\tau_1(a_0) = \tau_1(c_0)$ so $a_0 = c_0$. Then also $c_0 = a_0 = b_0$. Then $f(x)-f(y) = x-y$, $f(x)-f(z) = x-z$ whence $|f(x)-f(y)| \le |f(x)-f(z)|$.

Let $X \subset K$. If $f \in M_s(X)$ then $f^{-1} \in M_b(f(X))$. Conversely, if $f \in M_b(X)$ and $g : f(X) \to X$ is such that $f \circ g$ is the identity on $f(X)$ then $g \in M_s(f(X))$. This "asymmetry" can be "solved" in two ways.

DEFINITION 2.11 <u>Let</u> $X \subset K$ <u>and</u> $f : X \to K$. f <u>is called weakly monotone</u> ($f \in M_w(X)$) <u>if for all</u> $x,y,z \in X$

$$|x-y| < |x-z| \to |f(x)-f(y)| \le |f(x)-f(z)|$$

f <u>is called strongly monotone</u> ($f \in M_{bs}(X)$) if $f \in M_s(X) \cap M_b(X)$.

Clearly, $f \in M_{bs}(X)$ if and only if $f^{-1} \in M_{bs}(f(X))$. Also, if $f \in M_w(X)$ and $g : f(X) \to X$ is such that $f \circ g$ is the identity on $f(X)$ we have $g \in M_w(f(X))$.

Obviously we have $M_b(X) \cup M_s(X) \subset M_w(X)$ and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of M_w-functions, not for the sake of M_w itself but in order to gets results that are valid for M_b, M_s simultaneously. The functions in M_{bs} behave reasonable and they may be viewd as the non-archimedean equivalents of strict monotone functions in the real case.

THEOREM 2.12 Let $X \subset K$ and $f : X \to K$. Then the following conditions are equivalent.

(α) $f \in M_{bs}(X)$.

(β) f is injective and $C \mapsto f(C)$ is a 1-1 correspondence between the relatively convex subsets of X and those of $f(X)$.

(γ) For all $x,y,z \in X : |x-y| < |x-z| \leftrightarrow |f(x)-f(y)| < |f(x)-f(z)|$.

(δ) For all $x,y,z \in X : |x-y| = |x-z| \leftrightarrow |f(x)-f(y)| = |f(x)-f(z)|$.

(ε) For all $x,y,z \in X$ $|x-y| \leq |x-z| \leftrightarrow |f(x)-f(y)| \leq |f(x)-f(z)|$.

(ζ) $f \in M_s(X)$, $f^{-1} \in M_s(f(X))$.

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An M_w-function that is not in $M_s \cup M_b$). Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be any function, constant on the cosets of $\{x \in \mathbb{Z}_p : |x| < 1\}$. Then $f \in M_w(\mathbb{Z}_p)$. Clearly $f \notin M_s(\mathbb{Z}_p)$. $f \in M_b(\mathbb{Z}_p)$ if and only if the points of $f(\mathbb{Z}_p)$ are equidistant

(2) (Continuity of monotone functions). Let $X \subset K$.

(a) Let $f \in M_w(X)$. If $f(X)$ has no isolated points, then f is continuous. (Proof. Let $a \in X$ and $\varepsilon > 0$. Then there is $z \in X$ such that $z \neq a$, $|f(z)-f(a)| < \varepsilon$. Let $\delta := |z-a|$. Then for all $x \in X$ with $|x-a| < \delta$ we have, by the weak monotony of f, $|f(x)-f(a)| \leq |f(z)-f(a)| < \varepsilon$). It follows that if X and Y do not have isolated points and if f is an M_w-bijection of X onto Y, then f is a homeomorphism of X onto Y.

Conversely, it is easy to construct homeomorphisms of $\mathbb{Z}_p$ that are not in $M_w(\mathbb{Z}_p)$.

(b) If K is a local field then every $f \in M_w(X)$ is continuous. (See 5.1 (i)).

(c) If K has discrete valuation then every $f \in M_s(X)$ is continuous. (Example 2.4 (5) shows that such a statement is not true for $f \in M_b(X)$.) (Proof. If f were not continuous at some $a \in X$ then there would be an $\varepsilon > 0$ such that for some sequence $x_1, x_2, \ldots$ converging to a we had $|f(x_n)-f(a)| \geq \varepsilon$. We may suppose that $|x_1-a| > |x_2-a| > \ldots$. Since the valuation is discrete we have $\lim_{n\to\infty} |f(x_n)-f(a)| = 0$, a contradiction.)

(d) In 5.14 we shall give an example of a function in $M_{bs}(K)$ that is not continuous. (Of course, K will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" $M_w, \ldots$-conditions. Thus, by definition, $f \in M_{uw}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \rightarrow |f(x)-f(y)| \leq |f(z)-f(t)|$$

$f \in M_{us}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \rightarrow |f(x)-f(y)| < |f(z)-f(t)|$$

$f \in M_{ubs}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \leftrightarrow |f(x)-f(y)| < |f(z)-f(t)|.$$

Notice that $f \in M_{ubs}(X)$ means that $|f(x)-f(y)|$ is a strictly increasing function of $|x-y|$. Examples of such functions are isometries, but also the function $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ defined via

$$\Sigma a_n p^n \mapsto \Sigma a_n p^{2n} \qquad (\Sigma a_n p^n \in \mathbb{Z}_p)$$

$(|f(x)-f(y)| = |x-y|^2$ for all $x,y \in \mathbb{Z}_p$.)

Monotone functions : $\mathbb{R} \rightarrow \mathbb{R}$ are divided into two classes: the

increasing functions and the decreasing functions. For the non-archimedean case we may ask for a similar classification. First we try to express the situation in the real case in such a way that it can be translated. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be strictly monotone. If x runs through some side of a then $f(x)$ runs through some fixed side of $f(a)$. So there is a map $\sigma : \{-1,1\} \to \{-1,1\}$ such that $\sigma(\mathrm{sgn}(x-a)) = \mathrm{sgn}(f(x)-f(a))$ $(x \neq a)$. Apparently, the only σ's that can occur are the identity and $\sigma(x) = -x$ $(x \in \{1,-1\})$. Moreover it turns out that the map σ is independent of the choice of a.
The two maps σ that can occur can be interpreted as multiplication maps (with 1 and -1 respectively) or as the bijections $\{1,1\} \to \{-1,1\}$ and there seems to be no philosophical reason to make any decision of preference.

As an example, let us consider a function $f \in M_s(K)$. Let $a \in K$, let $\alpha \in \Sigma$. If $x \in a+\alpha$ and $y \in a+\alpha$ ("x,y are at the same side of a") then $x-a$, $y-a \in \alpha$, so $|x-y| < |y-a|$. Since $f \in M_s(K)$ we have $|f(x)-f(y)| < |f(y)-f(a)|$, whence $|f(x)-f(a)-(f(y)-f(a))| < |f(y)-f(a)|$, so $f(x)-f(a)$ and $f(y)-f(a)$ have the same sign. We may conclude that there is a map $\alpha_a : \Sigma \to \Sigma$ such that for all $x \in K$

$$x \in a+\alpha \to f(x) \subset f(a)+\sigma_a(\alpha) \qquad (\alpha \in \Sigma).$$

Unfortunately, it turns out that in general σ_a may be different from σ_b if $a \neq b$, even for isometrical maps. For example, let $p \neq 2$ and let τ be a permutation of $\{0,1,2,\ldots,p-1\}$ and define $f : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$\Sigma a_n p^n \mapsto \Sigma \tau(a_n) p^n \qquad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n).$$

Suppose we had a $\sigma : \Sigma \to \Sigma$ such that for all $x,y \in \mathbb{Z}_p$, $x-y \in \alpha$ implies $f(x)-f(y) \in \sigma(\alpha)$. Let $\alpha = \theta^i p^n$ (see 1.5). Then $x-y \in \alpha$ means

$x = a_0+a_1p+\ldots+a_np^n+\ldots$

$y = b_0+b_1p+\ldots+b_np^n+\ldots$

where $a_0 = b_0,\ldots,a_{n-1} = b_{n-1}$, $a_n-b_n = \theta^i$ modulo p.

Then $f(x)-f(y) = (\tau(a_n)-\tau(b_n))p^n+\ldots$, so $\sigma(\alpha) = \theta^j p^n$ where $\tau(a_n)-\tau(b_n) = \theta^j$ mod p. (j depending on i and n).

It turns out that $\tau(1)-\tau(0) = \tau(2)-\tau(1) =\ldots= \tau(p-1)-\tau(p-2)$ and it is clear that we can choose τ such that $\tau(1)-\tau(0) \neq \tau(2)-\tau(1)$, a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

DEFINITION 2.14 _Let_ $X \subset K$, $f : X \to K$ _and let_ $\sigma : \Sigma \to \Sigma$. _We say that_ f _is monotone of type_ σ _if for all_ $\alpha \in \Sigma$ _and all_ $x,y \in X$

$x-y \in \alpha$ _implies_ $f(x)-f(y) \in \sigma(\alpha)$.

(_In other words if_ $x >_\alpha y$ _implies_ $f(x) >_{\sigma(\alpha)} f(y)$, _see_ 1.3.)

(Notice that if $X \neq K$ f can be of type σ and of type $\tau : \Sigma \to \Sigma$ where $\sigma \neq \tau$, due to the fact that for some α, $x >_\alpha y$ for no $x,y \in X$, but we leave this technical fact aside for the moment.)

With our previous philosoply about real functions in mind, we define

DEFINITION 2.15 _Let_ $X \subset K$, $f : X \to K$, $\beta \in \Sigma$. _We say that_ f _is monotone of type_ β if for all $\alpha \in \Sigma$ _and all_ $x,y \in X$

$x-y \in \alpha$ _implies_ $f(x)-f(y) \in \alpha\beta$.

In other words, f is monotone of type β iff it is monotone of type σ

where $\sigma : \Sigma \to \Sigma$ is the multiplication with β. Or, equivalently, f is monotone of type β iff the sign of $\frac{f(x)-f(y)}{x-y}$ is constant β for all $x,y \in X$, $x \neq y$. This leads to

DEFINITION 2.16 *Let* $X \subset K$, $f : X \to K$. f *is called increasing if* f *is monotone of type* 1. *In other words,* f *is increasing if for all* $x,y \in X$, $x \neq y$ *the difference quotient* $\frac{f(x)-f(y)}{x-y}$ *is positive, i.e., if*

$$\left|\frac{f(x)-f(y)}{x-y} - 1\right| < 1.$$

In the next section we shall study the monotone functions of type σ and we will give a partial answer to the question for which maps $\sigma : \Sigma \to \Sigma$ there exists an $f : K \to K$ that is monotone of type σ.

3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE σ.

DEFINITION 3.1. Let $X \subset K$, $f: X \to K$. Let $\Phi f(x,y) := \dfrac{f(x)-f(y)}{x-y}$ $(x,y \in X,$ $x \neq y)$. *f is called*

positive if $f(X) \subset K^+$

strictly positive if $\sup_{x \in X} |f(x)-1| < 1$

increasing if $\Phi f(x,y) \in K^+$ *for all* $x,y \in X$, $x \neq y$

strictly increasing if $\sup\{|1-\Phi f(x,y)| : x,y \in X, x \neq y\} < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. *Let* $X \subset K$.

(i) *If* $f: X \to K$ *is (strictly) increasing and* $a \in K^+$ *then* af *is (strictly) increasing.*

(ii) *For* $a,b \in K$ *the function* $x \mapsto ax + b$ *is increasing if and only if* $a \in K^+$ *(if and only if* $x \mapsto ax + b$ *is strictly increasing).*

(iii) *If* $f: X \to K$ *is (strictly) increasing and* f *is (strictly) positive then* $-\frac{1}{f}$ *is (strictly) increasing.*

(iv) *The (strictly) increasing functions* $X \to K$ *form a convex set.*

(v) *If* $f: X \to K$ *and* $g: f(X) \to K$ *are (strictly) increasing then so is* $g \circ f$.

(vi) *If* $f: X \to K$ *is (strictly) increasing then so is* $f^{-1}: f(X) \to K$.

(vii) *If* $f_1, f_2, \ldots : X \to K$ *are increasing and* $f := \lim f_n$ *pointwise then* f *is increasing.*

(viii) *If* K *has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.*

Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function

$$\exp x = 1 + x + \frac{x^2}{2!} + \ldots$$

defined on $\{x \in K : |x| < p^{\frac{1}{1-p}}\}$ if $\chi(k) = p$, $X(K) = 0$ and on $\{x \in K : |x| < 1\}$ if $\chi(k) = 0$, is *increasing, as an easy computation may show*. In Section 2 we have seen that not every isometry is increasing.

(2) Let $f: X \to K$ be a C^1-function (i.e., Φf can continuously be extended to a function on $X \times X$, assume that $X \subset K$ has no isolated points. See [2]) and suppose $f'(a) \in K^+$ for some $a \in X$. Then f is locally (strictly) increasing at a.
(Proof. There is $\delta > 0$ such that $|x-a| < \delta$, $|y-a| < \delta$, $x \neq y$ implies

$$\left|\frac{f(x)-f(y)}{x-y} - f'(a)\right| \leq \tfrac{1}{2}.$$

For such x,y we have $\left|\frac{f(x)-f(y)}{x-y} - 1\right| \leq \left|\frac{f(x)-f(y)}{x-y} - f'(a)\right| \vee |f'(a)-1| \leq$ $\leq \max(\tfrac{1}{2}, |f'(a)-1|) < 1$.)

(3) The space $C(\mathbb{Z}_p)$ *of all continuous functions* $\mathbb{Z}_p \to \mathbb{Q}_p$, *is a* Banach space with respect to the sup norm $\|\ \|_\infty$. Let $e_0 := \xi_{\mathbb{Z}_p}$ and for $n \geq 1$ let $e_n := \xi_{B_n}$ where $B_n := \{x \in \mathbb{Z}_p : |x-n| < \frac{1}{n}\}$. It is proved in [1] that $e_0, e_1, \ldots$ is an orthonormal base of $C(\mathbb{Z}_p)$ i.e., for each $f \in C(\mathbb{Z}_p)$ there exists a unique null sequence $\lambda_0, \lambda_1, \ldots$ such that

$$f = \sum_{n=0}^{\infty} \lambda_n e_n$$

$$||f||_\infty = \max|\lambda_i|.$$

The coefficients λ_n can be reconstructed from f via

$$\lambda_0 = f(0)$$

$$\lambda_n = f(n)-f(n_-) \qquad (n \in \mathbb{N})$$

where n_- is defined as $a_0 + a_1p + \ldots + a_{s-1}p^{s-1}$ if $n = a_0 + a_1p + \ldots + a_sp^s$ $(a_s \neq 0)$ in base p.

Our aim is here to describe a necessary and sufficient condition for the λ_n in order that $f = \Sigma\, \lambda_n e_n$ is increasing. We show

$f = \Sigma\, \lambda_n e_n$ <u>is increasing if and only if for all</u> $n \in \mathbb{N}$

$$|\lambda_n-(n-n_-)| < |n-n_-|.$$

Proof. First observe that f is increasing if and only if for all $x \in \mathbb{Z}_p$

$$f(x) = x + g(x)$$

where $|\Phi g(x,y)| < 1$ for all $x,y \in \mathbb{Z}_p$, $x \neq y$.

As

$$x = \sum_{n\geq 1} (n-n_-)e_n(x) \qquad (x \in \mathbb{Z}_p)$$

it suffices to show that for $g = \Sigma\, \lambda_n e_n \in C(\mathbb{Z}_p)$ we have $|\Phi(g)| < 1$ if and only if $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Suppose first $|\Phi(g)| < 1$. Then for all $n \in \mathbb{N}$, $\left|\frac{f(n)-f(n_-)}{n-n_-}\right| < 1$, so $|\lambda_n| = |f(n)-f(n_-)| < |n-n_-|$.

Conversely, let $g = \Sigma\, \lambda_n e_n$ and let $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Let $x,y \in \mathbb{Z}_p$ and let $|x-y| = p^{-k}$ for some $k \in \{0,1,2,\ldots\}$. Since $e_n(a) = e_n(b)$ if and only if $|a-b| < \frac{1}{n}$ we have

$$e_n(x) = e_n(y) \quad \text{for } n < p^k.$$

Therefore

$$|g(x)-g(y)| = |\sum_{n\geq 1} \lambda_n (e_n(x)-e_n(y))| = |\sum_{n\geq p^k} \lambda_n (e_n(x)-e_n(y))|$$

$$\leq \max_{n\geq p^k} |\lambda_n| < \max_{n\geq p^k} |n-n_-| = p^{-k} = |x-y|$$

so $|\Phi g| < 1$.

(4) Let K have dense valuation and let k be (countably) infinite. Let X be the unit ball of K and let B_i ($i\in\mathbb{N}$) be the balls in X with radius 1^-. Choose $c_1, c_2, \ldots \in K$ such that $|c_1| < |c_2| < \ldots$, $\lim|c_n| = 1$. For $n \in \mathbb{N}$ define a function $f_n\colon X \to K$ via

$$f_n(x) = \begin{cases} x + c_i & \text{if } x \in B_i \quad (1\leq i\leq n) \\ x & \text{elsewhere} \end{cases}$$

Then each f_n is strictly increasing ($|\Phi f_n(x,y)-1| \leq \max\limits_{1\leq i,j\leq n} |c_i-c_j| \leq$ $\leq |c_n| < 1$). The sequence $f_1, f_2, \ldots$ converges pointwise to an increasing function f. But f is not strictly increasing:

$$\sup_{x\neq y}|\Phi f(x,y)-1| = \sup_{i,j}|c_i-c_j| = 1.$$

(Compare 3.2, (vii) and (viii).)

Increasing functions are closely related to functions g for which $|g(x)-g(y)| < |x-y|$ ($x \neq y$) (if f is increasing, set $g(x) := f(x)-x$).

DEFINITION 3.4. <u>Let</u> (X,ρ) <u>be an ultrametric space. A map</u> $g\colon X \to X$ <u>is called a pseudocontraction if</u> $\rho(f(x),f(y)) < \rho(x,y)$ $(x,y \in X,\ x \neq y)$.

The Banach contraction theorem states that X is complete if and only if each contraction $X \to X$ has a fix point. We have

LEMMA 3.5. _Let_ (X,ρ) _be an ultrametric space. Then the following conditions are equivalent._

(α) X _is spherically complete._

(β) _Each pseudocontraction_ $X \to X$ _has a fix point._

(γ) _Each pseudocontraction_ $X \to X$ _has a unique fix point._

Proof. If $\sigma: X \to X$ is a pseudocontraction and if x,y are fix points and $x \neq y$, then $\rho(x,y) = \rho(\sigma(x),\sigma(y)) < \rho(x,y)$, a contradiction. Thus, we have $(\beta) \Leftrightarrow (\gamma)$. We prove $(\alpha) \to (\beta)$. Let $B \subset X$ be a ball (i.e., either $B = \{x \in X: \rho(x,a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X: \rho(x,a) < r\}$ for some $a \in X$, $r > 0$). We call B _invariant_ if $\sigma(B) \subset B$. Now we observe two facts

a) X has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X: \rho(x,\sigma(a)) < \rho(a,\sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x),\sigma(a)) < \rho(x,a) \leq \max(\rho(x,\sigma(a)),\rho(\sigma(a),a)) = \rho(a,\sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \notin V$.

b) If B_1 and B_2 are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x,y)$ does not depend on x,y, since for $z \in B_1$, $u \in B_2$ $\rho(x,z) < \rho(x,y)$ and $\rho(y,u) < \rho(x,y)$, so $\rho(z,u) = \rho(x,y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x),\sigma(y)) < \rho(x,y)$, a contradiction).

It follows that the collection of invariant balls in X form a non-empty nest and by the spherical completeness of X there is a smallest invariant ball S. If $a \in S$, $\sigma(a) \neq a$ then $\{x \in S : \rho(x,\sigma(a)) < \rho(a,\sigma(a))\}$ is invariant and does not contain a, a contradiction. Hence, σ has a fix point (actually, S is a singleton).

We prove $(\beta) \rightarrow (\alpha)$. If X were not spherically complete, there exist balls $B_1 \supsetneq B_2 \supsetneq \dots$ such that $\bigcap_n B_n = \emptyset$. Choose $x_n \in B_n \backslash B_{n+1}$ $(n \in \mathbb{N})$, set $B_0 := X$ and define

$$\sigma(x) := x_{n+1} \text{ if } x \in B_n \backslash B_{n+1} \quad (n \in \{0,1,2,\dots\}).$$

Then σ has obviously no fix point. Let $x \in B_n \backslash B_{n+1}$ and $y \in B_m \backslash B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in B_{m+1}, whereas $x \in B_n \subset B_{m+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x),\sigma(y)) < \rho(x,y)$. Then σ is a pseudocontraction without a fix point. Contradiction.

COROLLARY 3.6. *The following conditions are equivalent.*

(α) K *is spherically complete*.

(β) *If* $C \subset K$ *is convex*, $f: C \rightarrow C$ *is increasing then* f *is surjective*.

(γ) *If* $C \subset K$ *is convex*, $f: C \rightarrow K$ *is increasing then* $f(C)$ *is convex*.

(δ) *An increasing* $f: K \rightarrow K$ *is surjective*.

Proof. $(\alpha) \rightarrow (\beta)$. Choose $a \in C$ and consider the map $\sigma: x \mapsto x - f(x) + a$ $(x \in C)$. Then $\sigma: C \rightarrow C$. C is spherically complete, σ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: f is surjective.

$(\beta) \rightarrow (\gamma)$. For a suitable $s \in K$, $f-s$ sends C into C. $(\gamma) \rightarrow (\delta)$ is clear.

$(\delta) \rightarrow (\alpha)$. Let $\sigma: K \rightarrow K$ be a pseudocontraction. Then $x \mapsto x - \sigma(x)$

is increasing hence is surjective. So then is $x \in K$ for which $x-\sigma(x) = 0$, i.e., σ has a fix point. By 3.5, K is spherically complete.

In case f is strictly increasing we do noet have to require that K is spherically complete:

THEOREM 3.7. <u>Let</u> $C \subset K$ <u>be convex and let</u> $f: C \to K$ <u>be strictly increasing. Then</u> $f(C)$ <u>is convex. If</u> $f(C) \subset C$, <u>then</u> $f(C) = C$.

<u>Proof</u>. Reread the proof of $(\alpha) \to (\beta)$, $(\beta) \to (\gamma)$ above. σ now is a contraction. C is complete. Apply the Banach contraction theorem.

Let X be a subset of $\mathbb{R}$ and let $f: X \to \mathbb{R}$ be a bounded increasing function. Then f can be extended to an increasing function $\mathbb{R} \to \mathbb{R}$ by setting $f(x) := \inf f$ if $x < y$ for all $y \in X$ and $f(x) := \sup\{f(y) : y \le x, y \in X\}$ for all other $x \in \mathbb{R}$. In our situation we can prove

THEOREM 3.8. The following conditions are equivalent.

(α) K <u>is spherically complete</u>.

(β) <u>For every</u> $X \subset K$ <u>an increasing function</u> $f: X \to K$ <u>can be extended to an increasing</u> $\bar{f}: K \to K$.

(γ) <u>Let</u> $X \subset K$, <u>and let</u> $f: X \to K$ <u>be a strictly increasing function. Then</u> f <u>can be extended to a strictly increasing function</u> $\bar{f}: K \to K$ <u>such that</u>

$$\sup_{\substack{x,y\in K\\ x\neq y}} \left|\frac{\bar{f}(x)-\bar{f}(y)}{x-y} - 1\right| = \sup_{\substack{x,y\in X\\ x\neq y}} \left|\frac{f(x)-f(y)}{x-y} - 1\right|.$$

<u>Proof</u>. $(\alpha) \to (\beta)$. Let $a \notin X$. By Zorn's Lemma it suffices to define $\bar{f}$ such that $\bar{f}$ is increasing on $X \cup \{a\}$. We are done if we can find $\alpha \in K$ such that for $x \in X$

$$\left|\frac{\alpha-f(x)}{a-x}-1\right|<1$$

i.e., $\alpha \in B_x := B_{f(x)-(a-x)}(|a-x|^-)$ $(x \in X)$.

Now $B_x \cap B_y \neq \emptyset$ $(x,y \in X)$ since the distance of their centers is $|f(x)-(a-x)-f(y)-(a-y)| = |f(x)-f(y)-(x-y)| = |\Phi f(x,y)-1||x-y| <$ $< \max(|x-a|,|a-y|)$. So if, say, $|x-a| \leq |y-a|$ we see that $|f(x)-(a-x)-$ $f(y)-(a-y)| < |y-a|$ whence $f(x)-(a-x) \in B_y$. By the spherical completeness of K we have $\bigcap_{x\in X} B_x \neq \emptyset$. Choose $\alpha \in \bigcap_{x\in X} B_x$.

$(\beta) \to (\alpha)$. Suppose K is not spherically complete. By 3.6, $(\delta) \to (\alpha)$ there is a non surjective increasing function $f: K \to K$. Then its inverse $g: f(K) \to K$ is increasing, surjective, and can obviously not be extended to an increasing $\bar{g}: K \to K$.

$(\beta) \leftrightarrow (\gamma)$ follows from the fact that (with $\mathbb{X}(x) = x$ for all x)

$$f \mapsto (1-c)\mathbb{X} + cf \qquad (c\in K,\ |c|<1)$$

is a 1-1 correspondence between the collection of all increasing functions on a set X and the collection of all strictly increasing functions g for which $|1-\Phi(g)| < |c|$.

We will now investigate the relation between increasingness of f and positivity of f'. First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry $\sigma: K \to K$. Let $\lambda \in K$, $0 < |\lambda| < 1$. Then $x \mapsto x-\lambda\sigma(x)$ is increasing, nowhere differentiable.

Clearly, if f is an increasing function, defined on some subset X of K without isolated points and if f is differentiable then for each

$x \in X$. $f'(x) = \lim_{y \to x} \Phi f(x,y) \in K^+$. So f' is positive. If, addition, f is strictly increasing, then f' is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let $f: X \to K$ be a (strictly) positive Baire class 1 function. Then does f have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

LEMMA 3.9. Let $X \subset K$ and let $f: X \to K$ be a Baire class 1 function such that $|f(x)| < 1$ for all $x \in X$. Then there exist locally constant functions $g_1, g_2, \ldots : X \to K$ such that $|g_n| \le 1 - \frac{1}{n}$ for each n and

$$f = \Sigma\, g_n \quad \text{(pointwise)}.$$

Proof. There exist continuous functions $f_1, f_2, \ldots : X \to K$ such that $f = \lim f_n$ pointwise. There exist locally constant functions $h_1, h_2, \ldots :$ $X \to K$ such that $|f_n - h_n| \le 2^{-n}$, hence $f = \lim h_n$ pointwise. Define $t_1, t_2, \ldots : X \to K$ as follows

$$t_n(x) := \begin{cases} h_n(x) & \text{if } |h_n(x)| \le 1 - \frac{1}{n} \\ 0 & \text{if } |h_n(x)| > 1 - \frac{1}{n} . \end{cases}$$

Then t_n is locally constant for each $n \in \mathbb{N}$. ($\{x \in X : |h_n(x)| \le 1 - \frac{1}{n}\}$ is closed and open in X). $|t_n| \le 1 - \frac{1}{n}$ and $\lim t_n = f$. Now let $g_1 := t_1$, and $g_n := t_n - t_{n-1}$ $(n \ge 2)$. Then $|g_1| = |t_1| = 0$, each g_n is locally constant, $|g_n| \le \max(|t_n|, |t_{n-1}|) \le \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n}$, $f = t_1 + \sum_{n=1}^{\infty} (t_{n+1} - t_n) = \sum_{n=1}^{\infty} g_n$.

LEMMA 3.10. *Let $X \subset K$ have no isolated points and let $f: X \to K$ be a Baire class 1 function, $|f(x)| < 1$ for all $x \in X$. Then f has an antiderivative F for which*

$$\left|\frac{F(x)-F(y)}{x-y}\right| < 1 \qquad (x,y \in X,\ x \neq y).$$

Proof. By Lemma 3.9, $f = \sum_{n=1}^{\infty} f_n$, where each f_n is locally constant, $|f_n| \leq 1 - \frac{1}{n}$. By [2] 3.9 each f_n has an antiderivative F_n for which

$$|F_n(x)-F_n(y)| \leq \max\ (|f_n(x)|,\ \tfrac{1}{2n})\,|x-y| \qquad (x,y \in X).$$

By [2] 3.7, $F := \sum F_n$ is an antiderivative of $\sum f_n = f$. Now for $x,y \in X$, $x \neq y$:

$$|F(x)-F(y)| \leq \sup_n |F_n(x)-F_n(y)| \leq \sup_n \max\ (|f_n(x)|, \tfrac{1}{2n})\,|x-y|$$

$\leq |x-y| \max_n\ (|f_n(x)|, \frac{1}{2})$. Now for each $x \in X$, $|f_n(x)| < 1$ for each n and $\lim |f_n(x)| = 0 < 1$. Hence $\max_n |f_n(x)| < 1$. It follows that $|F(x)-F(y)| < |x-y|$.

THEOREM 3.11. *Let $X \subset K$ have no isolated points and let $f: X \to K$ be (strictly) positive. Then f has a (strictly) increasing antiderivative.*

Proof. The function $x \mapsto f(x)-1$ has, by 3.10, an antiderivative H such that $|\Phi(H)| < 1$. Let $F(x): x + H(x)$ $(x \in X)$. Then $F' = f$ and $\Phi(F) = 1+\Phi(H)$. Thus, if f is positive then F is increasing. If f is strictly positive then $|f(x)-1| < r < 1$ for all $x \in X$ and, by a trivial extension of 3.10, we may choose H such that $|\Phi(H)| < r$. It follows that $|\Phi(F)-1| < r$, so F is strictly increasing.

We collect the results in

COROLLARY 3.12. *Let $X \subset K$ have no isolated points. Then*

(i) *If $f: X \to K$ is differentiable and (strictly) increasing then f' is a (strictly) positive Baire class 1 function.*

(ii) *If $g: X \to K$ is a (strictly) positive Baire class 1 function then g has a (strictly) increasing antiderivative.*

(iii) *If $f: X \to K$ is differentiable and if f' is (strictly) positive then $f = g+h$ where g is (strictly) increasing and where $h' = 0$.*

Note. We cannot strengthen 3.12 (iii) by replacing "$h' = 0$" by "h is locally constant". In fact, if $X = \mathbb{Z}_p$ then every locally constant function has bounded difference quotients. If our statements were true, then every differentiable $f: \mathbb{Z}_p \to \mathbb{Q}_p$ for which f' is positive would have bounded difference quotients.

But consider the function $f: \mathbb{Z}_p \to \mathbb{Q}_p$ defined via

$$f(x) := \begin{cases} x - p^{2n} & \text{if } |x-p^n| < p^{-3n} \quad (n \in \{0,1,2,\ldots\}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. Let $x_n := p^n$ and $y_n := p^n + p^{3n}$ ($n \in \mathbb{N}$). Then $f(x_n) = p^n - p^{2n}$, $f(y_n) = y_n = p^n + p^{3n}$, so $|f(x_n) - f(y_n)| = |p^{2n}| = p^{-2n}$, whereas $|x_n - y_n| = |p^{3n}| = p^{-3n}$. So

$$\lim_{n\to\infty} \left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right| = \lim_{n\to\infty} p^n = \infty.$$

We now study the connection between increasing C^1-functions and continuous positive functions.

If f is a (strictly) increasing C^1-function then clearly f' is a continuous (strictly) positive function.

Conversely, let $X \subset K$ have no isolated points and let $f\colon X \to K$ be continuous and positive. Let $P\colon C(X) \to C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \qquad (x \in X).$$

(Here the x_n are defined in the following way. Let $1 > r_1 > r_2 > \dots$, $\lim r_i = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of X into balls. Choose a center in each such ball. They form a collection R_n. We can arrange that $R_n \subset R_{n+1}$ for each n. For $n \in \mathbb{N}$, let $x_n := \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x)-x| < r_n$, $\sigma_n(x) \in R_n$. See [2] 5.3, 5.4.)

From [2] 5.4, it follows that Pf is a C^1-antiderivative of f. It suffices to prove that Pf is (strictly) increasing. Let $x,y \in X$, $x \neq y$, $|x-y| < r_1$. Then there is s such that $r_{s+1} \le |x-y| < r_s$. We have $x_1 = y_1, \dots, x_s = y_s$, $x_{s+1} \neq y_{s+1}$. Further $|x_{n+1}-x_n| \le |x-y|$ $(n>s)$, $|y_{n+1}-y_n| \le |x-y|$ $(n>s)$, $|x_{s+1}-y_{s+1}| \le |x-y|$. Hence (using the identity $x = \Sigma\,(x_{n+1}-x_n)+ x_1$, $y = \Sigma\,(y_{n+1}-y_n)+ y_1$, $x_1 = y_1$) $|Pf(x)-Pf(y)-(x-y)| =$

$$|(f(x_s)-1)(x_{s+1}-y_{s+1}) + \sum_{n>s} (f(x_n)-1)(x_{n+1}-x_n) - \sum_{n>s} (f(y_n)-1)(y_{n+1}-y_n)|.$$

If $|f(x)-1| < \alpha$ for all $x \in X$, we have since $\lim|f(x_n)-1|$ exists, $\sup_{n>s}|f(x_n)-1| < \alpha$, similarly, $\sup_{n>s}|f(y_n)-1| < \alpha$.
So we get $|Pf(x)-Pf(y)-(x-y)| < \alpha|x-y|$.

Now suppose $|x-y| \ge r_1$. Then since for all $n\colon |x_{n+1}-x_n| < r_1$, $|x_1-y_1| = |x-y|$ we get (again under the assumption $|f(x)-1| < \alpha$ for all $x \in X$):

$$|(Pf)(x)-(Pf)(y)-(x-y)| = |\sum_{n=1}^{\infty}(f(x_n)-1)(x_{n+1}-x_n) - \sum_{n=1}^{\infty}(f(y_n)-1)(y_{n+1}-y_n)| < |\alpha|r_1 \leq |\alpha|\,|x-y|.$$

We have proved:

THEOREM 3.13. <u>Let</u> $X \subset K$ <u>have no isolated points</u>. <u>Then the map</u> P <u>defined via</u>

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (f \in C(X),\ x \in X)$$

<u>maps (strictly) positive functions into (strictly) increasing functions</u>.

COROLLARY 3.14. <u>Let</u> $X \subset K$ <u>have no isolated points</u>. <u>Then if</u> $f \in C^1(X)$ <u>and</u> f' <u>is (strictly) positive, then</u> $f = j+h$ <u>where</u> j <u>is (strictly) increasing and</u> h <u>is locally constant</u>.

<u>Proof</u>. By 3.12 we have $f = j+h$ where j is (strictly) increasing and where $h' = 0$. Now by [2] Cor. 5.2^{bis} there is a locally constant function l: $X \to K$ with $\|\Phi(h-l)\|_\infty < \frac{1}{2}$. Then $s := j+(h-l)$ is (strictly) increasing, so we have $f = s+l$, where s is (strictly) increasing and l is locally constant.

<u>Note</u>. We may also define convex functions. Let $X \subset K$. A function $f: X \to K$ is called <u>convex</u> if the second order difference quotient is positive. I.e., if for all $x,y,z \in X$ $(x \neq y,\ y \neq z,\ x \neq z)$ we have

$$\Phi_2 f(x,y,z) := \frac{\Phi f(x,y)-\Phi f(x,z)}{y-z} = \frac{\frac{f(x)-f(y)}{x-y} - \frac{f(x)-f(z)}{x-z}}{y-z} \in K^+$$

(Just as we did for increasing function we may distinguish between convex and strictly convex.)

It follows that for a convex function f the function $x \mapsto \Phi f(x,y)$ defined on $X\setminus\{y\}$ is an isometry, hence can be continuously extended to the whole of X. Define $\overline{\Phi}f(y,y) = \lim_{x\to y} \Phi f(x,y)$ $(y \in X)$. Thus, f is differentiable. For all $x,y,z,t \in X$ we have

$$|\overline{\Phi}f(x,y)-\overline{\Phi}f(z,t)| \le \max(|\overline{\Phi}f(x,y)-\overline{\Phi}f(z,y)|,$$

$|\overline{\Phi}f(z,y)-\overline{\Phi}f(z,t)|) \le \max(|x-z|,|y-t|)$. Hence, $\overline{\Phi}f$ is uniformly continuous on X i.e., f is strongly uniformly differentiable in the sense of [2] page 67.

For each $y \in X$ the function $x \mapsto \Phi f(x,y)$ is increasing on X.
If $\chi(K) \neq 2$ then convexity of f implies increasingness of $\frac{1}{2}f'$.
(Proof.

$$\lim_{y\to x} \frac{\Phi f(x,y)-\Phi f(x',y)}{x-x'} = \frac{f'(x)-\Phi f(x',x)}{x-x'} \in K^+ \quad (x \neq x')$$

$$\lim_{y\to x'} \frac{\Phi f(x,y)-\Phi f(x',y)}{x-x'} = \frac{\Phi f(x,x')-f'(x')}{x-x'} \in K^+ \quad (x \neq x')$$

so $$\frac{f'(x)-f'(x')}{x-x'} \in 2K^+ \quad (x \neq x'), \text{ whence } \Phi(\tfrac{1}{2}f')(x,x') \in K^+ \text{ if } x \neq x'.)$$

Ofcourse, if $f \in C^2(X)$ (see [2] 8.1) then convexity of f implies positivity of D_2f ([2] 8.4). So if $\chi(K) \neq 2$ then $\frac{1}{2}f'' = D_2f$ ([2] 8.14) is positive. If $\chi(K) = 2$ then $f'' = 0$ for all C^2-functions.

Note. The functions that are monotone of type β ($\beta \in \Sigma$), see Def. 2.15, are easy to discribe: f is monotone of type β if and only if $b^{-1}f$ is increasing for any $b \in \beta$.

We now turn to the functions $X \to K$ that are of type σ where $\sigma : \Sigma \to \Sigma$. (2.14). For examples of such f, where σ is not a multiplier see 3.19 and 3.20. To avoid needless complications <u>from now on in this section we will assume that</u> X <u>is an open convex subset of</u> K. This implies that the set $\{\alpha \in \Sigma : \text{there is } x \in X \text{ such that } x \underset{\alpha}{>} a\}$ is independent of $a \in X$. Thus, X is homogeneous in the sense that $(a+\alpha) \cap X \neq \emptyset$ for some $a \in X$, $\alpha \in \Sigma$ then for each $b \in X$, $(b+\alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{\alpha \in \Sigma : \text{there is } x,y \in X \text{ such that } x \underset{\alpha}{>} y\}$. Then for each $a \in X$

$$\Sigma(X) = \{\alpha \in \Sigma : x \underset{\alpha}{>} a \text{ for some } x \in X\}.$$

Either $\Sigma(X) = K$ or $\Sigma(X) = \{\alpha \in \Sigma : |\alpha| < r\}$ for some $r > 0$ or $\Sigma(X) = \{\alpha \in \Sigma : |\alpha| \leq r\}$ for some $r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see 1.2) i.e., if $\alpha,\beta \in \Sigma(X)$ and $\alpha \oplus \beta$ is defined then $\alpha \oplus \beta \in \Sigma(X)$.

To be sure that f is monotone both of type σ and type τ implies $\sigma = \tau$ we define

DEFINITION 3.15. (<u>Let</u> $X \subset K$ <u>be open</u>, <u>convex and</u>) <u>let</u> $\sigma : \Sigma(X) \to \Sigma$. $f : X \to K$ <u>is called monotone of type</u> σ <u>if for all</u> $x,y \in X$ <u>and</u> $\alpha \in \Sigma(X)$

$$x \underset{\alpha}{>} y \to f(x) \underset{\sigma(\alpha)}{>} f(y).$$

THEOREM 3.16. <u>Let</u> $f : X \to K$ <u>be monotone of type</u> $\sigma : \Sigma(X) \to \Sigma$. <u>Then</u>

(i) $\sigma(-\alpha) = -\sigma(\alpha) \quad (\alpha \in \Sigma(X))$.

(ii) <u>Let</u> $\alpha,\beta \in \Sigma(X)$. <u>If</u> $\sigma(\alpha) \oplus \sigma(\beta)$ <u>is defined then so is</u> $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$.

(iii) <u>Let</u> $\alpha,\beta \in \Sigma(X)$. <u>If</u> $|\alpha| < |\beta|$ <u>then</u> $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) <u>Let</u> s <u>be in the prime field of</u> K <u>and let</u> $|s| = 1$. <u>Then</u> $\sigma(s\alpha) = s\sigma(\alpha) \quad (\alpha \in \Sigma(X))$.

(v) <u>If</u> $\beta \in \Sigma(X)$, $|\beta| = 1$, β <u>contains an element of the prime field of</u> K <u>then</u> $\sigma(\beta\alpha) = \beta\sigma(\alpha)$ <u>for all</u> $\alpha \in \Sigma(X)$.

(vi) $f \in M_{us}(X)$ (*i.e.*, *for all* $x,y,z,t \in X$, $|x-y| < |z-t|$ *implies* $|f(x)-f(y)| < |f(z)-f(t)|$).

(vii) f *is either nowhere continuous or uniformly continuous on* X.

Proof.

(i) Let $x,y \in X$ such that $x \underset{\alpha}{>} y$. Then $f(x)-f(y) \in \sigma(\alpha)$; $f(y)-f(x) \in -\sigma(\alpha)$. But also $y \underset{-\alpha}{>} x$, hence $f(y)-f(x) \in \sigma(-\alpha)$. So $-\sigma(\alpha)$ and $\sigma(-\alpha)$ are not disjoint and they must coincide.

(ii) Suppose $\sigma(\alpha) \oplus \sigma(\beta)$ is defined. If $\alpha \oplus \beta$ were not, then $\beta = -\alpha$ so, by (i), $\sigma(\beta) = \sigma(-\alpha) = -\sigma(\alpha)$. Hence also $\alpha \oplus \beta$ is defined. Choose $x,y \in X$ with $x \underset{\alpha}{>} y$. There is $z \in X$ such that $y \underset{\beta}{>} z$. Then $x-y \in \alpha$, $y-z \in \beta$, so $x-z \in \alpha \oplus \beta$. Further $f(x)-f(y) \in \sigma(\alpha)$, $f(y)-f(z) \in \sigma(\beta)$ so $f(x)-f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also $x-z \in \alpha \oplus \beta$, so $f(x)-f(z) \in \sigma(\alpha \oplus \beta)$. The signs $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ are not disjoint and they must coincide.

(iii) Let $|\alpha| < |\beta|$. Choose x,y,z such that $x-y \in \alpha$, $y-z \in \beta$. Then (see 1.2 and preamble) $f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(\alpha)+\sigma(\beta)$, $x-z \in \alpha + \beta = \alpha \oplus \beta = \beta$, so $f(x)-f(z) \in \sigma(\beta)$. Thus $[\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta)$ is not empty. If $\sigma(\alpha) \oplus \sigma(\beta)$ were not defined then $\sigma(\alpha) = -\sigma(\beta)$ and $\sigma(\alpha) + \sigma(\beta)$ would be a ball with center 0 and radius $|\sigma(\beta)|^{-}$, but then $[\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta)$ would be empty. Hence $\sigma(\alpha) \oplus \sigma(\beta)$ is defined and by (ii) we have $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. By (1.2) (vi), $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) Let $\chi(K) \neq 0$. Then $s = n\cdot 1$ for some $n \in \{1,2,\dots,\chi(K)-1\}$, so by 1.2 (vii), $s\alpha = n\alpha = \underset{n}{\oplus}\, \alpha$, $s\sigma(\alpha) = n\sigma(\alpha) = \underset{n}{\oplus}\, \sigma(\alpha)$. By a repeated application of (ii), we see $\sigma(\underset{n}{\oplus}\, \alpha) = \underset{n}{\oplus}\, \sigma(\alpha)$. Hence $\sigma(s\alpha) = s\sigma(\alpha)$.

Let $\chi(k) = 0$. Let s be of the form $n\cdot 1$ for some $n \in \mathbb{N}$. By a similar reasoning as above, $\sigma(s\alpha) = s\sigma(\alpha)$. We may identify the prime field of K with $\mathbb{Q}$.

Now observe that $\{s \in K^*: \sigma(s\alpha) = s\sigma(\alpha)$ for all $\alpha \in \Sigma\}$ is a multiplicative subgroup of K^* containing all elements of the form $n\cdot 1$ $(n\in\mathbb{N})$, and -1 (by (i)), hence it must contain $\mathbb{Q}^*$.

Let $\chi(K) = 0$, $\chi(k) = p \neq 0$. By a similar reasoning as above, we arrive at the fact that $\{s \in K^*: \sigma(s\alpha) = s\sigma(\alpha)$ for all $\alpha \in \Sigma\}$ is a multiplicative subgroup of K^* containing $-1,1,2,\dots,p-1$. We may identify the prime field of K with $\mathbb{Q}$. If $n \in \mathbb{N}$, $n \equiv s \bmod p$ $(1\leq s\leq p-1)$ then $n\alpha = s\alpha$ for all α, so $\sigma(n\alpha) = \sigma(s\alpha) = s\sigma(\alpha) = n\sigma(\alpha)$. Thus our multiplicative group contains all $n \in \mathbb{Z}$ that ar not divisible by p, so it contains all $s \in \mathbb{Q}$, having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If $|x-y| < |z-t|$ and if $x-y \neq 0$ then $x-y \in \alpha$, $z-t \in \beta$ for some $\alpha,\beta \in \Sigma$, $|\alpha| < |\beta|$. By (iii) $|\sigma(\alpha)| < |\sigma(\beta)|$ so $|f(x)-f(y)| < |f(z)-f(t)|$. For the case $x-y = 0$ we must prove that f is injective. Now $z \neq t$, so $z-t \in \alpha$ for some α hence $f(z)-f(t) \in \sigma(\alpha)$. Thus, $f(z) \neq f(t)$.

(vii) Let $\rho := \inf_{x\neq y}|f(x)-f(y)|$. If $\rho > 0$ then clearly f is nowhere continuous. If $\rho = 0$, let $\varepsilon > 0$. There is $a,b \in X$, $a \neq b$ such that $|f(a)-f(b)| < \varepsilon$. By (vi), for all $x,y \in X$ with $|x-y| < |a-b|$, $|f(x)-f(y)| < |f(a)-f(b)| < \varepsilon$. Then f is uniformly continuous.

A natural question that can be raised is the following. If $f : X \to K$ is of type σ, does it follow that σ is injective? We have (see also 3.18 and 3.20)

THEOREM 3.17. <u>Let</u> $f : X \to K$ <u>be monotone of type</u> σ. <u>Then the following conditions are equivalent.</u>

(α) σ <u>is injective</u>.

(β) $f \in M_b(X)$.

(γ) $f \in M_{ubs}(X)$.

(δ) <u>If, for</u> $\alpha,\beta \in \Sigma(X)$, $\alpha \oplus \beta$ <u>is defined then so is</u> $\sigma(\alpha) \oplus \sigma(\beta)$ (and $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$).

(ε) <u>If</u> $\alpha,\beta \in \Sigma(X)$, $|\sigma(\alpha)| < |\sigma(\beta)|$ <u>then</u> $|\alpha| < |\beta|$.

<u>Proof</u>. We prove (α) $\to$ (ε) $\to$ (γ) $\to$ (β) $\to$ (δ) $\to$ (α).

(α) $\to$ (ε). Let $|\sigma(\alpha)| < |\sigma(\beta)|$ then $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$ (1.2.(vi)). By 3.16, (iii), $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. Since σ is injective, $\alpha \oplus \beta = \beta$ so (again 1.2.(vi)) $|\alpha| < |\beta|$.

(ε) $\to$ (γ). Let $|x-y| \leq |z-t|$ ($x,y,z,t \in X$). We prove $|f(x)-f(y)| \leq |f(z)-f(t)|$. If $z = t$ there is nothing to prove. Assume $z \neq t$ and $|f(x)-f(y)| > |f(z)-f(t)|$. Then ($f$ is injective), supposing $x-y \in \alpha$, $z-t \in \beta$ for some $\alpha,\beta \in \Sigma(X)$, we have $f(x)-f(y) \in \sigma(\alpha)$, $f(z)-f(t) \in \sigma(\beta)$ and $|\sigma(\alpha)| > |\sigma(\beta)|$. By ($\varepsilon$), $|\alpha| > |\beta|$ i.e., $|x-y| > |z-t|$. Contradiction.

(γ) $\to$ (β). Trivial.

(β) $\to$ (δ). Suppose $\sigma(\alpha) \oplus \sigma(\beta)$ is not defined. Then $|\sigma(\alpha)| = |\sigma(\beta)|$ and, by 3.16 (iii), $|\alpha| = |\beta|$. Choose x,y,z such that $x-y \in \alpha$, $y-z \in \beta$. Then $f(x)-f(z) \in \sigma(\alpha) + \sigma(\beta)$ so $|f(x)-f(z)| < |\sigma(\alpha)| = |f(x)-f(y)|$. Since $f \in M_b(X)$, $|x-z| < |x-y|$ hence, since $x-z \in \alpha \oplus \beta$, $x-y \in \alpha$: $|\alpha \oplus \beta| < |\alpha|$. But $|\alpha \oplus \beta| = \max(|\alpha|,|\beta|)$, a contradiction.

(δ) $\to$ (α). Suppose $\sigma(\alpha) = \sigma(\beta)$ and $\alpha \neq \beta$. Then $\alpha \oplus (-\beta)$ is defined. By (δ), also $\sigma(\alpha) \oplus \sigma(-\beta)$ is defined. But $\sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha)$, so $\sigma(\alpha) \oplus -\sigma(\alpha)$ is defined, a contradiction.

THEOREM 3.18. <u>Let</u> k <u>be a prime field. Then, if</u> $f : X \to K$ <u>is monotone of type</u> σ <u>then</u> σ <u>is injective</u>.

<u>Proof</u>. Suppose $\sigma(\alpha) = \sigma(\beta)$ for some $\alpha,\beta \in \Sigma(X)$. Then $|\sigma(\alpha)| = |\sigma(\beta)|$ so, by 3.16 (iii), $|\alpha| = |\beta|$. There is $t \in K$, $|t| = 1$ such that $\beta = t\alpha$. Since k is a prime field we may suppose $t \in \{1,2,\ldots,p-1\}$ if $k \cong \mathbb{F}_p$ and $t \in \mathbb{Q}^*$ if $k \cong \mathbb{Q}$. So, by 3.16 (iv), $\sigma(\beta) = \sigma(t\alpha) = t\sigma(\alpha) = t\sigma(\beta)$. For $x \in \sigma(\beta)$ we have $tx \in \sigma(\beta)$, so $tx \cdot x^{-1} \in K^+$ i.e., $|t-1| < 1$. It follows easily that $t = 1$. Hence, $\alpha = \beta$.

We now like to determine all $\sigma : \Sigma \to \Sigma$ that "can occur" as the type of a monotone function in case $K = \mathbb{Q}_p$. We use the fact that Σ can be identified with the following subgroup of $\mathbb{Q}_p^*$

$$\{\theta^i p^n : i \in \{0,1,2,\ldots,p-2\},\ n \in \mathbb{Z}\}$$

where θ is a primitive $(p-1)^{\text{th}}$ root of 1. (See 1.5.)

First, let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be monotone of some type $\sigma : \Sigma \to \Sigma$. By 3.18, σ is injective. By 3.17, (ε), 3.16 (iii) we have $|\alpha| < |\beta| \leftrightarrow |\sigma(\alpha)| < |\sigma(\beta)|$ and $|\alpha| = |\beta| \leftrightarrow |\sigma(\alpha)| = |\sigma(\beta)|$, so $|\sigma(\alpha)|$ is a strictly increasing function of $|\alpha|$.

Set

$$\sigma(\theta^i p^n) = \theta^{s(i,n)} p^{\lambda(i,n)} \qquad (\theta^i p^n \in \Sigma)$$

Where $s : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \mathbb{Z}$. We see that $|\sigma(\theta^i p^n)| = |\sigma(\theta^j p^n)|$ for all $i,j \in \{0,1,2,\ldots,p-2\}$ hence $\lambda(i,n) = \lambda(j,n)$ for all $i,j \in \{0,1,2,\ldots,p-2\}$. Then

$$\sigma(\theta^i p^n) = \theta^{s(i,n)} p^{\lambda(n)}$$

where $\lambda : \mathbb{Z} \to \mathbb{Z}$ is a strictly increasing function (in the classical sense).

By 3.16 (v), $\sigma(\theta^i p^n) = \theta^i \sigma(p^n) = \theta^i \theta^{s(0,n)} p^{\lambda(n)}$.

Thus, σ is of the form

$$(*)\quad \theta^i p^n \to \theta^i \theta^{s(n)} p^{\lambda(n)}$$

where $s : \mathbb{N} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we are given a map σ of the form $(*)$ then it is easy to construct an $f : \mathbb{Q}_p \to \mathbb{Q}_p$, monotone of type σ. In fact, let $x \in \mathbb{Q}_p$, $x = \sum_{n\in\mathbb{Z}} a_n p^n$, where $a_n \in \{0,1,\theta,\ldots,\theta^{p-2}\}$ for each n and $a_{-n} = 0$ for large n. Then set

$$f(x) := \sum_{n\in\mathbb{Z}} a_n \theta^{s(n)} p^{\lambda(n)}.$$

Now let $x = \sum a_n p^n$, $y = \sum b_n p^n$ and $\pi(x-y) = \theta^i p^m$ for some $i \in \{0,1,\ldots,p-2\}$, $m \in \mathbb{Z}$. Then $a_n = b_n$ for $n < m$ and $a_m - b_m = \theta^i \bmod p$. So the sign of $a_m - b_m$ is θ^i. $f(x)-f(y) = \sum_{n\geq m} (a_n - b_n)\theta^{s(n)} p^{\lambda(n)} = (a_m - b_m)\theta^{s(m)} p^{\lambda(m)} + r$, where $|r| < |f(x)-f(y)|$. The sign of $f(x)-f(y)$ is the sign of $(a_m-b_m)\theta^{s(m)}p^{\lambda(m)}$ which is $\theta^i\theta^{s(m)}p^{\lambda(m)}$. So $\pi(f(x)-f(y)) = \theta^i\theta^{s(m)}p^{\lambda(m)} = \sigma(\theta^i p^m)$. Thus, f is monotone of type σ. We have found

THEOREM 3.19. *The set $\{\sigma : \Sigma \to \Sigma$: there is $f : \mathbb{Q}_p \to \mathbb{Q}_p$, monotone of type $\sigma\}$ is equal to the set of all $\sigma : \Sigma \to \Sigma$ of the form*

$$\theta^i p^n \mapsto \theta^i \theta^{s(n)} p^{\lambda(n)}$$

where $s : \mathbb{Z} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Remark. With the notations as in 3.19, let $\mu(n) := \lambda(n) - n$. Then $\mu : \mathbb{Z} \to \mathbb{Z}$ is increasing $(\mu(n+1) = \lambda(n+1)-(n+1) \geq \lambda(n)+1-(n+1) = \mu(n))$. We then have two possibilities for a function $f : \mathbb{Q}_p \to \mathbb{Q}_p$, monotone of type σ.

(a) $\lim_{n\to\infty} \mu(n) = \infty$. Then $|\sigma(\alpha)| = |\alpha||p^{\mu(n)}|$, $(\alpha = \theta^i p^n)$, so $\lim_{|\alpha|\to 0} |\frac{\sigma(\alpha)}{\alpha}| = 0$. Thus $\lim_{x-y\to 0} |\frac{f(x)-f(y)}{x-y}| = 0$ uniformly: f <u>is uniformly differentiable and</u> $f' = 0$.

(b) μ is bounded above. Then $\mu(n)$ is constant, c, for $n \geq n_0$. (For example, if σ is bijective then we have even $\mu(n) = c$ for all n.) Thus, for sufficiently small $|\alpha|$ $(\alpha = \theta^i p^n \in \Sigma)$ we have

$$|\sigma(\alpha)| = |p^{\lambda(n)}| = |p^{n+c}| = |p^c||\alpha|.$$

So, there is r such that $|x-y| < r$ implies $|f(x)-f(y)| = |p^c||x-y|$. In this case we then have: <u>there is</u> $r > 0$ <u>and</u> $\lambda \in \mathbb{Q}_p$ <u>such that on each ball in</u> $\mathbb{Q}_p$ <u>of radius</u> r, $\lambda^{-1}f$ <u>is an isometry</u>.

We now construct an example of a function f monotone of type σ, wher σ is not injective. Let $p = 3 \bmod 4$ and let $K := \mathbb{Q}_p(\sqrt{-1})$. The elements of K can be written as $a+bi$ $(a,b \in \mathbb{Q}_p)$ and $|a+bi| = \max(|a|,|b|)$. The value group of K is the same as the one of $\mathbb{Q}_p$, the residue class field has p^2 elements (hence is not a prime field, see 3.18). Let X be the unit ball of K, let

$$S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}.$$

For each $x \in X$ there is a unique $\bar{x} \in S$ such that $|x-\bar{x}| < 1$. As in section 1, let $\pi : K^* \to \Sigma$ be the sign map. Notice that $\pi(s) \neq \pi(t)$ for $s,t \in S^*$, $s \neq t$.

Define a function $h : S \to K$ as follows

$$h(a+bi) = \frac{1}{p} a \quad (a+bi \in S)$$

and let $f : X \to K$ be defined via

$$f(x) = x + h(\overline{x}) \qquad (x \in X).$$

We claim that f is monotone of type σ where

$$\sigma(\pi(a+bi)) = \pi(\frac{1}{p} a) \text{ if } a+bi \in S,\ a \neq 0$$

$$\sigma(\alpha) = \alpha \qquad \text{elsewhere.}$$

(Clearly, σ is a well defined map $\Sigma(X) \to K$, σ is not injective since, for example, $\sigma(\pi(1)) = \sigma(\pi(1+i))$.

<u>Proof</u>. Let $|\alpha| < 1$ and $x-y \in \alpha$, then $|x-y| < 1$ so $\overline{x} = \overline{y}$, $h(\overline{x}) = h(\overline{y})$. It follows that $f(x)-f(y) = x-y \in \alpha = \sigma(\alpha)$.

Now let $|\alpha| = 1$ be of the form $\pi(bi)$, $b \in \{1,2,\dots,p-1\}$ and let $x-y \in \alpha$. Say, $\overline{x} = r+si$, $\overline{y} = t+ui$ $(r,s,t,u \in \{0,1,2,\dots,p-1\})$. Then also $\overline{x}-\overline{y} \in \alpha$, so $|r+si-t-ui-bi| < 1$ hence $r = t$. Thus, $h(\overline{x}) = \frac{1}{p} r = h(\overline{y})$, and we have $f(x)-f(y) = x-y \in \alpha = \sigma(\alpha)$.

Finally, let $|\alpha| = 1$, $\alpha = \pi(a+bi)$, where $a \neq 0$ $(a,b \in \{0,1,2,\dots,p-1\})$ and let $x-y \in \alpha$. Set $\overline{x} = r+si$, $\overline{y} = t+ui$. Then $\overline{x}-\overline{y} \in \alpha$, so $r-t = a \bmod p$. We find $h(\overline{x}) = \frac{1}{p} r$, $h(\overline{y}) = \frac{1}{p} t$, so $|h(\overline{x})-h(\overline{y}) - \frac{1}{p} a| < \frac{1}{|p|}|a|$ i.e. $h(\overline{x})-h(\overline{y}) \in \pi(\frac{1}{p} a)$. Since $|\pi(x-y)| \le 1$, we find $f(x)-f(y) = x-y-(h(\overline{x})-h(\overline{y})) \in \pi(\frac{1}{p} a) = \sigma(\pi(a+bi)) = \sigma(\alpha)$.

Concluding:

EXAMPLE 3.20. <u>Let</u> $p = 3 \bmod 4$ <u>and</u> $K = \mathbb{Q}_p(\sqrt{-1})$. <u>Then there exists a function</u> $f : \{x \in K : |x| \le 1\} \to K$, <u>monotone of some type</u> σ, <u>where</u> σ <u>is not injective</u>.

In case K has discrete valuation we have some extra information.

THEOREM 3.21. *Let* K *have discrete valuation and let* $f : X \to K$ *be monotone of type* $\sigma \in \Sigma(X)$. *Then*

(i) f *is continuous.*

(ii) *If* σ *is injective, then* f *and* $\Phi(f)$ *are bounded on bounded sets.*

(iii) *If* σ *is injective and if* $f(X)$ *is convex then there is* $\lambda \in K$ *such that* λf *is an isometry.*

Proof. (i) Follows from 3.16 (vi) and 2.13 (2)(c). If σ is injective then by 3.16 (iii) and 3.17 (ε), $|\sigma(\alpha)|$ is a strictly increasing function of $|\alpha|$. Suppose X is bounded. Then $\Sigma(X) = \{\alpha \in \Sigma : |\alpha| \le r\}$ for some $r \in K^*$. Let $|\sigma(\alpha)| = s$ whenever $|\alpha| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|\alpha| = |\pi| r$. Then $|\alpha| < r$ so $|\sigma(\alpha)| < s$, hence $|\sigma(\alpha)| \le |\pi| s$. By induction, it follows that $|\sigma(\alpha)| \le |\pi|^n s$ whenever $|\alpha| = |\pi|^n r$, so $|\sigma(\alpha)| \le |\alpha| \cdot \frac{s}{r}$. So the difference quotients of f are bounded by sr^{-1}. Then clearly f is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $\Sigma(f(X))$ has the form $\{\alpha \in \Sigma : |\alpha| \le s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then σ induces an injection of $\{\rho \in |K^*| : \rho \le r\}$ onto $\{\alpha \in |K^*| : |\alpha| \le s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(\alpha)| = c|\alpha|$ for some $c \in \mathbb{R}$ i.e., $|f(x)-f(y)| = |c||x-y|$ for all $x,y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type σ, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) K is a local field (f is continuous).

(2) k is finite and $X = \{x \in K : |x| \le 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be

representatives modulo $\{x \in K : |x| < 1\}$, let $M = \max|f(a_i)-f(a_j)|$. For each $x,y \in X$ we have i,j for which $|x-a_i| < 1$, $|y-a_j| < 1$. Since $f \in M_s(X)$, we have $|f(x)-f(a_i)| < M$, $|f(y)-f(a_j)| < M$ whence $|f(x)-f(y)| \leq M$: f is bounded.)

(3) K is discrete, σ is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let k be isomorphic to the algebraic closure of $\mathbb{F}_p$. Let X be the unit ball of K. Then there exists a function $f : X \to K$, monotone of type σ, for some $\sigma : \Sigma(X) \to \Sigma$ such that

(i) σ is not injective.

(ii) f, $\Phi(f)$ are unbounded.

Proof.

As an $\mathbb{F}_p$-vector space, k has a countable base $e_1, e_2, \ldots$. For any $\lambda \in \mathbb{F}_p$, $\lambda = n1_{\mathbb{F}_p}$ for some $n \in \{0,1,2,\ldots,p-1\}$. (Here for a field L, 1_L is the unit element of L.) Define $\tilde{\lambda} := n1_K$. Choose $c_1, c_2, \ldots \in K$ such that $1 < |c_1| < |c_2| < \ldots, \lim_{n\to\infty}|c_n| = \infty$, and define a map $h : k \to K$ via

$$h(\Sigma\,\lambda_n e_n) = \Sigma\,\tilde{\lambda}_n c_n \qquad (\Sigma\,\lambda_n e_n \in k)$$

Define $f : X \to K$ by

$$f(x) = x + h(\overline{x}) \qquad (x \in X)$$

(Here $\overline{x}$ is the image of x under the canonical map $X \to k$).

Then clearly f is unbounded and so is $\Phi(f)$.

Let us identify $\{\alpha \in \Sigma : |\alpha| = 1\}$ with k^* in the obvious way. We claim that f is monotone of type σ where

$$\sigma(\alpha) = \begin{cases} \alpha \text{ if } |\alpha| < 1 \\ \pi(\widetilde{\lambda}_n c_n) \text{ if } \alpha = \Sigma\, \lambda_m e_m,\ n = \max\{m : \lambda_m \neq 0\}. \end{cases}$$

In fact, let $x-y \in \alpha$ and $|\alpha| < 1$. Then $h(\bar{x}) = h(\bar{y})$ so $f(x)-f(y) = x-y \in \sigma(\alpha)$. Now let $x-y \in \alpha$ where $|\alpha| = 1$. Then set $\bar{x} = \Sigma\, \lambda_n e_n$, $\bar{y} = \Sigma\, \mu_n e_n$. Let $r = \max\{n : \lambda_n \neq \mu_n\}$. Then $\bar{x}-\bar{y} = \sum_{n=1}^{r} (\lambda_n - \mu_n) e_n = \alpha$, so $\sigma(\alpha) = \pi((\widetilde{\lambda_r - \mu_r}) c_r)$.

On the other hand, $f(x)-f(y) = x-y-(h(\bar{x})-h(\bar{y})) = x-y-\Sigma(\widetilde{\lambda}_n - \widetilde{\mu}_n)c_n = x-y-\sum_{n=1}^{r} (\widetilde{\lambda}_n - \widetilde{\mu}_n)c_n$. Thus $\pi(f(x)-f(y)) = \pi((\widetilde{\lambda}_r - \widetilde{\mu}_r)c_r)$.

Now we have $\widetilde{\lambda_r - \mu_r} = \widetilde{\lambda}_r - \widetilde{\mu}_r \bmod p$, so $\pi(\widetilde{\lambda_r - \mu_r}) = \pi(\widetilde{\lambda}_r - \widetilde{\mu}_r)$. It follows that $f(x)-f(y) \in \sigma(\alpha)$.

Obviously, σ is not injective.

We will consider briefly the differentiable functions, monotone of type σ. Let $f : X \to K$ be such a function. If $f'(a) = 0$ for some $a \in X$, then $\lim_{|\alpha| \to 0} \frac{|\sigma(\alpha)|}{|\alpha|} = 0$, so $f' = 0$ uniformly on X. Now let $f'(a) \neq 0$. Then if x is sufficiently close to a, we have $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < |f'(a)|$. Thus for $|\alpha|$ small enough we have $f'(a) \in \frac{\sigma(\alpha)}{\alpha}$ i.e. $\frac{\sigma(\alpha)}{\alpha}$ is constant. This implies that $\pi(f'(x))$ does not depend on x (f' has constant sign) and that locally f is monotone of type β for some $\beta \in \Sigma$.

We end this section with a discussion on the question which maps $\sigma : \Sigma \to \Sigma$ do occur as a type of a monotone function.

LEMMA 3.23. Let $\sigma : \Sigma \to \Sigma$. Suppose σ satisfies

if $\alpha \oplus \beta$ is defined then $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$. $(\alpha, \beta \in \Sigma)$.

Then

(i) $\sigma(-\alpha) = -\sigma(\alpha) \quad (\alpha \in \Sigma)$.

(ii) <u>If</u> $\sigma(\alpha)$ ... <u>is defined then so is</u> $\alpha \oplus \beta$.

(iii) <u>If</u> $|\alpha| < |\beta|$ <u>then</u> $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) σ <u>is injective</u>.

(v) <u>If</u> $|\alpha| = |\beta|$ <u>then</u> $|\sigma(\alpha)| = |\sigma(\beta)|$.

<u>Proof</u>. (i) is trivial if $\chi(k) = 2$, so suppose $\chi(k) \neq 2$ and let $-\sigma(\alpha) \neq \sigma(-\alpha)$ for some $\alpha \in \Sigma$. Then we have the identity $(\alpha \oplus \alpha) \oplus (-\alpha) = \alpha$, so $\sigma(\alpha \oplus \alpha) \oplus \sigma(-\alpha) = \sigma(\alpha)$, whence $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha) = \sigma(\alpha)$. Now by 1.2 (iii) $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ (this last expression is defined. If not, then $-\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(-\alpha)$. Now $\sigma(\alpha) \oplus \gamma = -\sigma(\alpha)$ has only one solution namly $\gamma = -2\sigma(\alpha)$. So we then would have $\sigma(-\alpha) = -2\sigma(\alpha) = -(\sigma(\alpha) \oplus \sigma(\alpha))$, but this contradicts the existence of $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha))$. From $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ we obtain by 1.2 (vi): $|\sigma(\alpha) \oplus \sigma(-\alpha)| < |\sigma(\alpha)|$. On the other hand, by 1.2 (v), $|\sigma(\alpha) \oplus \sigma(-\alpha)| = |\sigma(\alpha)| \vee |\sigma(-\alpha)|$. Contradiction. (i) follows.

Now (ii) follows easily from (i): if $\alpha \oplus \beta$ were not defined then $\beta = -\alpha$ so, by (i), $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, a contradiction. Let $|\alpha| < |\beta|$, then $\alpha \oplus \beta = \beta$, so $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. By 1.2 (vi) we find $|\sigma(\alpha)| < |\sigma(\beta)|$. We proved (iii).
If $\sigma(\alpha) = \sigma(\beta)$ and $\alpha \neq \beta$ then $\sigma(\alpha \oplus (-\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, an absurdity. So σ is injective (iv). Finally, let $|\alpha| = |\beta|$ and $|\sigma(\alpha)| > |\sigma(\beta)|$. Then $\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(\beta) =$ (by (ii)) $= \sigma(\alpha \oplus \beta)$. By injectivity of σ, $\alpha = \alpha \oplus \beta$, and by 1.2 (vi), we find $|\beta| < |\alpha|$.

Now we have

LEMMA 3.24. *Let* K *be spherically complete, let* $Y \subset K$ *(not necessarily convex) and let* $\tau : \Sigma(Y)$ $(= \{\pi(x-y) : x,y \in Y,\ x \neq y\}) \to \Sigma$ *such that* f *is monotone of type* τ *(i.e.,* $x,y \in Y$, $x-y \in \alpha \in \Sigma(Y)$ *then* $f(x)-f(y) \in \tau(\alpha)$).
Suppose τ *can be extended to a* $\sigma : \Sigma \to \Sigma$ *satisfying the condition of Lemma* 3.23. *Then* f *can be extended to a monotone function* $\overline{f} : K \to K$ *of type* σ.

Proof. By Zorn's lemma, it suffices to extend f to $Y \cup \{a\}$ $(a \neq Y)$ such that $f(x)-\overline{f}(a) \in \sigma(\pi(x-a))$, $\overline{f}(a)-f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x))$ $(x \in Y)$. Each B_x is a ball with radius $|\sigma(\pi(a-x))|^-$. By the spherical completeness of K, we are done if we can show that $B_x \cap B_y \neq \emptyset$ $(x \neq y,\ x,y \in Y)$.
Set $\alpha := \pi(a-x)$ and $\beta := \pi(a-y)$. Let $b \in \sigma(\alpha)$; $c \in \sigma(\beta)$. We prove: $|f(x)+b-f(y)-c| < |\sigma(\alpha)| \vee |\sigma(\beta)|$. We have two cases:
1) $\alpha = \beta$. Then $a-x \in \alpha$, $a-y \in \alpha$ implies $|x-y| < |a-x| = |\alpha|$, so $|\pi(x-y)| < |\alpha|$ whence $|\pi(f(x)-f(y))| = |\sigma(\pi(x-y))| < |\sigma(\alpha)|$ (by 3.23 (iii)), so $|f(x)-f(y)| < |\sigma(\alpha)|$. Further, $b \in \sigma(\alpha)$, $c \in \sigma(\alpha)$ implies $|b-c| < |\sigma(\alpha)|$, hence $|f(x)+b-f(y)-c| < |\sigma(\alpha)|$.
2) $\alpha \neq \beta$. Then $x-y = a-y-(a-x) \in \beta \oplus (-\alpha)$, so $f(x)-f(y)+b-c \in \sigma(\beta \oplus -\alpha) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus -\alpha) + \sigma(\alpha \oplus -\beta) = \sigma(\beta \oplus (-\alpha))-\sigma(\beta \oplus -\alpha)$, hence $|f(x)-f(y)+b-c| < |\sigma(\beta \oplus -\alpha)| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(\alpha)|,|\sigma(\beta)|)$.

THEOREM 3.25. *Let* K *be spherically complete and let* $\sigma : \Sigma \to \Sigma$. *Suppose*

$$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha,\beta \in \Sigma,\ \alpha \neq -\beta).$$

Then there exists a function $f : K \to K$, *monotone of type* σ.

Proof. Choose $Y := \{0\}$ and let $g : Y \to K$ be defined via $g(0) = 0$. Then g satisfies the conditions of Lemma 3.24 so it can be extended to a function f of type σ.

We now give a description of the maps $\sigma : \Sigma \to \Sigma$ mentioned in 3.23. For each $r \in |K^*|$ choose $\alpha_r \in \Sigma$ such that $|\alpha_r| = r$. Further, there is a natural isomorphim of multiplicative groups between k^* and $\{\alpha \in \Sigma : |\alpha| = 1\}$, denoted by $1 \mapsto \alpha_1$ $(1 \in k^*)$. Of course, if $1+1' \neq 0$ then $\alpha_{1+1'} = \alpha_1 \oplus \alpha_{1'}$. Each element of Σ can be written in only one way as $\alpha_r\alpha_1$ $(r \in |K^*|,\ 1 \in k^*)$. Now if σ is as in 3.23 we get

$$\sigma(\alpha_r\alpha_1) = \alpha_{\lambda(r)}\alpha_{n(r,1)}$$

where $\lambda : |K^*| \to |K^*|$ is strictly increasing and $1 \mapsto n(r,1)$ is an injective group endomorphism of the additive group k. Conversely, if $\lambda : |K^*| \to |K^*|$ is strictly increasing and for each r, $1 \mapsto n(r,1)$ is an injective group homomorphism $k \to k$ then

$$\alpha_r\alpha_1 \mapsto \alpha_{\lambda(r)}\alpha_{n(r,1)} \quad (\alpha_r\alpha_1 \in \Sigma)$$

satisfies the condition of 3.23. So we get

THEOREM 3.26. <u>Let</u> K <u>be spherically complete and let</u> $|K| = [0,\infty)$. <u>Then there exist a nowhere continuous</u> $f : K \to K$, <u>monotone of some type</u> $\sigma : \Sigma \to \Sigma$.

Proof. With the notations as above, let $\sigma : \Sigma \to \Sigma$ be defined as follows

$$\sigma(\alpha_r\alpha_1) = \alpha_{r+1}\alpha_1.$$

By 3.25 there is an $f : K \to K$ monotone of type σ. Clearly $|f(x)-f(y)| \geq 1$ if $x \neq y$ so f is nowhere continuous.

4. MONOTONE FUNCTIONS, GENERAL THEOREMS

In this section we study $M_w(X)$, $M_b(X)$, $M_s(X)$,... . To avoid unnecessary complications we ASSUME THROUGHOUT THIS SECTION THAT X IS A CLOSED SUBSET OF K WITHOUT ISOLATED POINTS. We collect here the results on monotone functions that are valid for general K. In the next section we will see what happens if we put some extra conditions on K (e.g., $|K|$ discrete, ...).

First two elementary lemmas.

LEMMA 4.1 *Let* $f : X \to K$. *Then the following conditions are equivalent*

(α) $f \in M_w(X)$ (see Def.2.11).

(β) *For all* $x,y,z \in X$, $|x-y| < |x-z|$ *implies* $|f(x)-f(z)| = |f(y)-f(z)|$.

(γ) *For all* $x,y,z \in X$, $|f(x)-f(z)| \neq |f(y)-f(z)|$ *implies* $|x-y| = \max(|x-z|,|y-z|)$.

Proof. (α) → (β). $|x-y| < |x-z|$ implies $|y-z| = |x-z| > |x-y|$, so

$$|f(x)-f(y)| \le \min(|f(x)-f(z)|,|f(y)-f(z)|).$$

It follows that $|f(x)-f(z)| = |f(y)-f(z)|$.

(β) → (γ). (β) says that $|f(x)-f(z)| \neq |f(y)-f(z)|$ implies $|x-y| \ge |x-z|$. By symmetry, also $|x-y| \ge |y-z|$ where $|x-y| \ge \max(|x-z|,|y-z|)$. The opposite inequality is trivial.

(γ) → (α). Let $|x-y| < |x-z|$. Then $|x-y| \neq \max(|x-z|,|z-y|)$ so, by (γ), $|f(x)-f(z)| = |f(y)-f(z)|$. Then $|f(x)-f(y)| \le \max(|f(x)-f(z)|,|f(y)-f(z)|) = |f(x)-f(z)|$.

LEMMA 4.2 (i) *If* $f \in M_w(X)$, $\lambda \in K$ *then* $\lambda f \in M_w(X)$.

(ii) <u>If</u> $f_1, f_2, \ldots \in M_w(X)$ <u>and</u> $f := \lim f_n$ <u>pointwise then</u> $f \in M_w(X)$.

(iii) <u>If</u> $f \in M_w(X)$ <u>and</u> $g : f(X) \to X$ <u>is such that</u> $f \circ g$ <u>is the identity on</u> $f(X)$, <u>then</u> $g \in M_w(f(X))$. <u>In particular, if</u> f <u>is injective and weakly monotone then so is</u> f^{-1}. (Notice that $f(X)$ need not be closed and may have isolated points.)

<u>Proof</u>. Obvious.

<u>Remark</u>. Lemmas, similar to 4.1 and 4.3, but now for $M_b(X)$, $M_s(X)$, $M_{bs}(X)$ have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an M_w-function need not be continuous (see 2.4(5), 3.26) we will derive properties of M_w-functions that are closely related to continuity.

LEMMA 4.3 <u>Let</u> $f \in M_w(X)$. <u>Then</u> f <u>is bounded on precompact subsets of</u> X.

<u>Proof</u>. Let $Y \subset X$ be precompact. Assume that Y is not a singleton. Then Y is bounded and has a positive diameter $r = \max\{|x-y| : x,y \in Y\}$. The equivalence relation $x \sim y$ iff $|x-y| < r$ divides Y into finitely many classes $Y_1, \ldots, Y_n$ $(n \geq 2)$. Choose $a_i \in Y_i$ for each i, and let $M := \max_{1 \leq i \leq n} |f(a_i)|$. We prove: $|f| \leq M$. In fact, let $x \in Y$. Then there is i such that $|x-a_i| < r$. Choose $j \neq i$. We have $|x-a_i| < |a_i-a_j|$ whence $|f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M$. So $|f(x)| \leq M$.

The following lemma shows that an $f \in M_w(X)$ at $a \in X$ is either continuous or "very discontinuous".

LEMMA 4.4 <u>Let</u> $f \in M_w(X)$ <u>and let</u> $a \in X$. <u>Then we have the following alternative</u>.

Either f is continuous at a, or for each sequence $x_1, x_2, \ldots \in X$ ($x_n \neq a$ *for all* n) *with* $\lim x_n = a$ *the sequence* $f(x_1), f(x_2), \ldots$ *is bounded and has no convergent subsequence.*

Proof. Since $\{x_1, x_2, \ldots\}$ is precompact the set $\{f(x_1), f(x_2), \ldots\}$ is bounded by Lemma 4.3. We are done if we can prove the following. If $x_1, x_2, \ldots$, $\lim x_n = a$, $x_n \neq a$ for all n, $\lim_{n\to\infty} f(x_n)$ exists, then f is continuous at a. Now set $\alpha := \lim f(x_n)$. Let $y_1, y_2, \ldots \in X$, $\lim y_n = a$. We prove $\lim f(y_n) = \alpha$. (Then it follows that $\alpha = f(a)$ since we may choose $y_n := a$ for all n.) Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|f(x_k)-\alpha| < \varepsilon$. For n sufficiently large we have $|y_n-a| < |x_k-a|$, so for large m (depending on n) we have $|y_n-x_m| < |x_k-x_m|$, whence $|f(y_n)-f(x_m)| \leq |f(x_k)-f(x_m)|$. Since $\lim_{m\to\infty} f(x_m) = \alpha$ we find $|f(y_n)-\alpha| \leq |f(x_k)-\alpha| < \varepsilon$, so $\lim_{n\to\infty} f(y_n) = \alpha$.

COROLLARY 4.5 *Let* $f \in M_w(X)$. *Then the graph of* f

$$\Gamma_f := \{(x,y) \in X \times K : y = f(x)\}$$

is closed in K^2.

Proof. Let $(x_n, f(x_n)) \in \Gamma_f$ and let $\lim x_n = x$, $\lim f(x_n) = \alpha$. If $x_n = x$ for infinitely many n then $\alpha = f(x)$, so $(x,\alpha) \in \Gamma_f$. If not then by the alternative of lemma 4.4, f is continuous at x, so $\alpha = f(x)$ and $(x,\alpha) \in \Gamma_f$.

COROLLARY 4.6 *Let* $f \in M_w(X)$. *If each bounded subset of* f(X) *is precompact then* f *is continuous*.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.

LEMMA 4.7 *Let* $f \in M_w(X)$ *and let* $Y \subset f(X)$ *be precompact. Then either* *f is constant on* $f^{-1}(Y)$ *or* $f^{-1}(Y)$ *is bounded.*

Proof. It suffices to prove: if $Z \subset X$ is unbounded and $f(Z)$ is precompact then f is constant on Z. Let $a,b \in Z$. Since Z is unbounded there are $x_1,x_2,\ldots \in Z$ such that

$$(*) \quad |a-b| < |x_1-a| < |x_2-a| < \ldots$$

Since $f(Z)$ is precompact we may assume (by taking a suitable subsequence) that $\alpha := \lim f(x_n)$ exists. From $(*)$ we obtain $|x_1-x_2| = |x_2-a|$, $|x_2-x_3| = |x_3-a|,\ldots$, so

$$|a-b| < |x_1-a| < |x_1-x_2| < |x_2-x_3| < \ldots$$

hence

$$|f(a)-f(b)| \leq |f(x_1)-f(a)| \leq |f(x_1)-f(x_2)| \leq \ldots$$

it follows that $|f(a)-f(b)| = \lim_{n\to\infty} |f(x_n)-f(x_{n+1})| = 0$ i.e., $f(a) = f(b)$.

LEMMA 4.8 *Let* $f \in M_w(X)$ *and let* $\alpha \in \overline{f(X)}$ *be a non-isolated point of* $\overline{f(X)}$. *Then we have the following alternative. Either*

I. *There is* $a \in X$ *such that for each sequence* $x_1,x_2,\ldots$ *in* X *for which* $\lim_{n\to\infty} f(x_n) = \alpha$ *we have* $\lim x_n = a$, *or*

II. *If* $x_1,x_2,\ldots \in X$, $\lim f(x_n) = \alpha$, $f(x_n) \neq \alpha$ *for all* n, *then* $x_1,x_2,\ldots$ *is bounded and has no convergent subsequence.*

Proof. Suppose we are not in case II. Since α is not isolated in $\overline{f(X)}$ and $f(X)$ is dense in $\overline{f(X)}$ we have a sequence $x_1,x_2,\ldots$ in X for which $f(x_n) \neq \alpha$ for each n, and $\lim_{n\to\infty} f(x_n) = \alpha$. Since f is not constant on $\{x_1,x_2,\ldots\}$ it follows by Lemma 4.7 that $\{x_1,x_2,\ldots\}$ is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by $x_1,x_2,\ldots$ and set

$a := \lim x_n$. Then $a \in X$. Now let $y_1, y_2, \ldots$ be a sequence in X for which $\lim f(y_n) = \alpha$. We prove that $\lim y_n = a$. In fact, let $\varepsilon > 0$. There is $k \in \mathbb{N}$ such that $|x_k - a| < \varepsilon$. For large n we have $|f(y_n)-\alpha| < |f(x_k)-\alpha|$, so for large m (depending on n) we have $|f(y_n)-f(x_m)| < |f(x_k)-f(x_m)|$ whence $|y_n - x_m| \le |x_k - x_m|$, so $|y_n - a| \le |x_k - a| < \varepsilon$.

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function $f : X \to K$ <u>injective at</u> $a \in X$ if $f(x) = f(a)$ for some $x \in X$ implies $x = a$.

Now suppose that we have $\alpha \in \overline{f(X)}$, not isolated, for which we are in alternative I. Then for a sequence $x_1, x_2, \ldots$ with $\lim f(x_n) = \alpha$ we have $\lim x_n = a \in X$ so $(a, \alpha) = \lim_{n\to\infty} (x_n, f(x_n))$, so by Cor.4.5 we have $\alpha = f(a)$. Thus, $\alpha \in f(X)$. f is injective at a: if $f(b) = f(a)$ then since $\lim f(b) = \alpha$ we must have $\lim b = a$ i.e. $b = a$. Further, f is continuous at a (see 2.13 (2)(a)).

If each bounded subset of X is precompact we never can be in case II. This is also true if $f \in M_b(X)$ and $|X|$ is discrete i.e. if $x_i, y_i \in X$ $|x_1 - y_1| > |x_2 - y_2| > \ldots$ then $\lim |x_n - y_n| = 0$. Proof: let $\alpha \in \overline{f(X)}$ and let $\lim f(x_n) = \alpha$, $f(x_n) \ne \alpha$ for all n. Without loss of generality we may assume

$$|\alpha - f(x_1)| > |\alpha - f(x_2)| > \ldots$$

hence
$$|f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots$$

and, since $f \in M_b(X)$

$$|x_1 - x_2| > |x_2 - x_3| > \ldots$$

Since $|X|$ is discrete, the sequence $x_1, x_2, \ldots$ is convergent. So we have case I. We find

THEOREM 4.9 *Let either each closed and bounded subset of* X *be compact and* $f \in M_w(X)$, *or let* $|X|$ *be discrete and* $f \in M_b(X)$. *Then*

(i) $f(X)$ *is closed (in* K*).*

(ii) *If* $f(a) \in f(X)$ *is not isolated then* f *is injective at* a, f *is continuous at* a. *In particular if* $f(X)$ *has no isolated points then* f *is a homeomorphism* $X \simeq f(X)$.

Proof. If $\alpha \in \overline{f(X)}\setminus f(X)$ then α is not isolated. Since we are in alternative I of Lemma 4.8, $\alpha \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an f satisfying the conditions of 4.9 maps closed subsets of X (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 *Let* $f : X \to K$.

(i) *If* $f \in M_w(X)$ *and if* $Y \subset X$ *is closed and the closed and bounded subsets of* Y *are compact then* $f(Y)$ *is spherically complete.*

(ii) *If* $f \in M_b(X)$ *and if* $Y \subset X$ *is spherically complete then so is* $f(Y)$.

(iii) *If* $f \in M_s(X)$ and if $A \subset f(X)$ *is spherically complete then so is* $f^{-1}(A)$.

Proof. (i) Let $B_1 \supsetneq B_2 \supsetneq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1\setminus B_2$, $f(y_2) \in B_2\setminus B_3, \ldots$ Then we have

$$|f(y_1)-f(y_2)| > |f(y_2)-f(y_3)| > \ldots$$

and, by the weak monotony of f

$$|y_1-y_2| \geq |y_2-y_3| \geq \dots$$

Suppose first that $\lim |y_n-y_{n+1}| = 0$. Then $y := \lim y_n$ and there are infinitely many k for which

$$|y_k-y_{k-1}| > |y_{k+1}-y_k|.$$

Now $|y-y_k| \leq \max(|y_k-y_{k+1}|,|y_{k+1}-y_{k+2}|,\dots) \leq |y_k-y_{k+1}|$. So we get for infinitely many k

$$|y-y_k| < |y_k-y_{k-1}|$$

whence $\qquad |f(y)-f(y_k)| \leq |f(y_k)-f(y_{k-1})|$

so $f(y) \in B_{k-1}$ for infinitely many k, i.e., $f(y) \in \bigcap_k B_k$.
Next, suppose that $|y_{k+1}-y_k| \geq \varepsilon > 0$ for all k. Then since $y_1,y_2,\dots$ is bounded it has a convergent subsequence $y_{n_1},y_{n_2},\dots$. Let $y := \lim y_{n_i}$. Then we have for infinitely many i

$$|y-y_{n_i}| < \varepsilon \leq |y_{n_i}-y_{n_i+1}|$$

whence $\qquad |f(y)-f(y_{n_i})| \leq |f(y_{n_i})-f(y_{n_i+1})|,$

so $f(y) \in B_{n_i}$ for infinitely many i i.e., $f(y) \in \bigcap_k B_k$.
(ii) Let $B_1 \supsetneq B_2 \supsetneq \dots$ be balls in $f(Y)$ and let $y_1,y_2,\dots \in Y$ such that $f(y_1) \in B_1\setminus B_2$, $f(y_2) \in B_2\setminus B_3,\dots$. Then we have

$$|f(y_1)-f(y_2)| > |f(y_2)-f(y_3)| > \dots$$

and since $f \in M_b(X)$:

$$|y_1-y_2| > |y_2-y_3| > \dots$$

Since Y is spherically complete, there is $y \in Y$ such that $|y-y_n| \leq |y_n-y_{n+1}|$ for all n, hence $|f(y)-f(y_n)| \leq |f(y_n)-f(y_{n+1})|$ for all n. It follows that $f(y) \in B_n$ for all n.
(iii) Let $B_1 \supsetneq B_2 \supsetneq \dots$ be balls in $f^{-1}(A)$. Choose $x_1 \in B_1\setminus B_2$, $x_2 \in B_2\setminus B_2,\dots$
Then $|x_1-x_2| > |x_2-x_3| > \dots$ whence $|f(x_1)-f(x_2)| > |f(x_2)-f(x_3)| > \dots$
There is $x \in f^{-1}(A)$ such that $|f(x)-f(x_n)| \leq |f(x_n)-f(x_{n+1})|$ for all n.
Hence $|x-x_n| \leq |x_n-x_{n+1}|$ for all n i.e., $x \in \bigcap_n B_n$.

DEFINITION 4.11 <u>Let</u> $f : X \to K$. <u>The oscillation function</u> $\omega_f : X \to [0,\infty]$ <u>is defined by</u>

$$\omega_f(a) := \lim_{n\to\infty} \sup\{|f(x)-f(y)| : |x-a| \le \frac{1}{n}, |y-a| \le \frac{1}{n}, x,y \in X\} \quad (a \in X)$$
$$= \lim_{n\to\infty} \sup\{|f(x)-f(a)| : |x-a| \le \frac{1}{n}, x \in X\}.$$

THEOREM 4.12 <u>Let</u> $f \in M_w(X)$. <u>Then</u>

$$\omega_f(a) = \inf_{z\neq a} |f(z)-f(a)| \qquad (a \in X).$$

<u>Proof</u>. *For $x \neq a$ we have $|f(x)-f(a)| \ge \inf_{z\neq a} |f(z)-f(a)|$ and (since a is not* isolated) consequently

$$\omega_f(a) \ge \inf_{z\neq a} |f(z)-f(a)|.$$

Conversely, let $z \neq a$. Then for all x such that $|x-a| < |z-a|$ we have

$$|f(x)-f(a)| \le |f(z)-f(a)|$$

so

$$\omega_f(a) \le |f(z)-f(a)|$$

whence

$$\omega_f(a) \le \inf_{z\neq a} |f(z)-f(a)|.$$

THEOREM 4.13 <u>Let</u> $f \in M_w(X)$, $a \in X$. <u>If</u> $x_1,x_2,\ldots \in X$, $\lim_{n\to\infty} x_n = a$ ($x_n \neq a$ for all n) <u>then</u> $\lim_{n\to\infty} |f(x_n)-f(a)| = \omega_f(a)$.

<u>Proof</u>. By 4.12 we have $\underline{\lim}_{n\to\infty} |f(x_n)-f(a)| \ge \omega_f(a)$. Conversely, $\overline{\lim}\, |f(x_n)-f(a)| \le \omega_f(a)$ is clear from the definition of ω_f.

5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: K is local, k is finite, K has discrete valuation. Also we can sometimes say a little more if we assume X to be convex. For the time being, let X be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case K is a local field.

THEOREM 5.1 *Let* K *be a local field, and let* $f \in M_W(X)$. *Then*

(i) f *is continuous*.

(ii) *If* $Y \subset X$ *is closed then* $f(Y)$ *is closed*.

(iii) *If* $f(X)$ *is bounded and* f *is not constant then* X *is bounded*.

(iv) *Let* $a \in X$. *Then the following are equivalent*

(α) f *is not injective at* a

(β) f *is locally constant at* a

(γ) $f(a)$ *is isolated in* $f(X)$.

(v) *The following conditions are equivalent*

(α) f *is injective*

(β) $f(X)$ *has no isolated points*

(γ) f *is a homeomorphism of* X *onto* $f(X)$.

Proof. Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for M_b- and M_s-functions in case X is convex and K is a local field. First, a lemma that is valid in a more general situation.

LEMMA 5.2 *Let the residue class field* k *of* K *be finite. Let* X *be convex and let* $f \in M_b(X)$. *Then*

(i) *If* $a,b,c \in X$, $|a-b| < |a-c|$, $f(a) \neq f(c)$ *then* $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) *If* $C \subset X$ *is convex then* $f(C)$ *is convex in* $f(X)$ (*f is weakly Darboux continuous, see* 2.5).

(iii) *If* f *is injective, then* $f \in M_s(X)$.

Proof. (i) Let $B := \{x \in K : |x-a| \leq |a-c|\}$. Then $B \subset X$ and $f(B) \subset [f(a),f(c)]$. Define an equivalence relation on B by: $x \sim y$ if $|f(x)-f(y)| < |f(a)-f(c)|$.

Since k is finite we get finitely many equivalence classes $B_1,B_2,\ldots,B_n$. Since $a \not\sim c$ we have $n \geq 2$. The diameter of $f(B)$ equals $|f(a)-f(c)|$, the distance between $f(B_i)$ and $f(B_j)$ equals $|f(a)-f(c)|$ $(i \neq j)$. Since $[f(a),f(c)]$ can contain at most $q := \chi(k)$ sets having distances $|f(a)-f(c)|$ to one another we have $n \leq q$. Hence $2 \leq n \leq q$. By 2.2 (β), each B_i is convex. If $x,y \in B_i$ and $|x-y|$ were $|a-c|$ then $B_i = B$, contradicting $n \geq 2$. Thus B is a disjoint union of n balls $B_1,\ldots,B_n$, where $2 \leq n \leq q$ and $|x-y| < |a-c|$ whenever $x,y \in B_i$ $(i = 1,\ldots,n)$. It follows that $n = q$ and that each B_i has the form $\{x \in K : |x-b_i| < |a-c|\}$ $(b_i \in B)$. Hence, if $|a-b| < |a-c|$ then there is i for which $a,b \in B_i$. So $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) Let $a,b \in C$ and let $\alpha \in f(X)$ with $\alpha \in [f(a),f(b)]$. We show that $\alpha \in f(C)$. If $f(a) = f(b)$ this is clear. If $f(a) \neq f(b)$, set $\alpha = f(x)$ where $x \in X$. Then $|f(x)-f(a)| \leq |f(b)-f(a)|$. If $|x-a|$ were $> |b-a|$ then $f(x) \neq f(a)$ (since $f \in M_b(X)$) and by (i) we then had $|f(b)-f(a)| < |f(x)-f(a)|$, a contradiction. Hence $|x-a| \leq |b-a|$ i.e., $x \in [a,b] \subset C$, so $\alpha = f(x) \in f(C)$.

(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on k, see 2.10.

COROLLARY 5.3 *Let K be a local field and let $f \in M_b(X)$ and X convex. Then the following conditions are equivalent.*

(α) $f \in M_s(X)$.

(β) f is injective.

(γ) $f \in M_{bs}(X)$.

(δ) $f(X)$ has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 *Let K be a local field and let X be the unit ball of K (or any bounded convex set, for that matter). If either $f \in M_s(X)$ or $f \in M_b(X)$ then f has bounded difference quotients.*

Proof. f is bounded, let $M := \sup\{|f(x)-f(y)| : x,y \in X\}$. It suffices to prove that $|f(x)-f(0)| \leq M|x|$ for all x. Let $\pi \in K$, $|\pi| < 1$, be a generator of the value group. By induction on n we prove:

$$\text{if } |x| = |\pi|^n \text{ then } |f(x)-f(0)| \leq |\pi|^n M.$$

The statement is clear for $n = 0$. Now suppose the statement is true for $0,1,\ldots,n-1$.

Let $x \in X$, $|\alpha| = |\pi|^n$. Then $|x-0| < |\pi^{n-1}-0|$. If $f(\pi^{n-1}) \neq f(0)$ we have either since $f \in M_s(X)$ or by 5.2(i)

$$|f(x)-f(0)| < |f(\pi^{n-1})-f(0)| \leq |\pi|^{n-1}M$$

hence

$$|f(x)-f(0)| \leq |\pi|^n M$$

If $f(\pi^{n-1}) = f(0)$ then $|f(x)-f(0)| \leq |f(\pi^{n-1})-f(0)| = 0$, so certainly $|f(x)-f(0)| \leq |\pi|^n M$.

Notes.

(a) 5.4 cannot be extended to the case $X = K$. In fact, let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be the map $\Sigma a_n p^n \mapsto \Sigma a_n p^{2n}$. ($\Sigma a_n p^n \in \mathbb{Q}_p$.) Then $f \in M_{bs}(\mathbb{Q}_p)$ but $|p^n f(p^{-n})| = p^n \to \infty$.

(b) If we loose the condition on K, for example by requiring that the valuation is discrete then 3.22 and 2.4(5) show that the conclusion of 5.4 is false both for M_b-functions and M_s-functions. On the other hand, it is clear from the proof of 5.4 that a bounded M_s-function on X has bounded difference quotients.

(c) One may wonder how difference quotients of M_w-functions behave. See the example below.

EXAMPLE 5.5 *Let $p \neq 2$. Then there is an $f \in M_w(\mathbb{Z}_p \to \mathbb{Q}_p)$ that has unbounded difference quotients.*

Proof. Let $a_0, a_1, \ldots$ be defined via $a_{2n} := p^n$ $(n = 0,1,2,\ldots)$ and $a_{2n+1} := 2p^n$ $(n = 0,1,2,\ldots)$. Thus $(a_0, a_1, a_2, \ldots) = (1,2,p,p^2,2p^2,\ldots)$. Then $|a_0| \geq |a_1| \geq |a_2| \geq \ldots$, $\lim a_n = 0$, $|a_n - a_m| = |a_m|$ $(n > m)$.

Set

$$f(x) := \begin{cases} a_n & \text{if } |x| = p^{-n} \quad (n = 0,1,2,\ldots) \\ 0 & \text{if } x = 0 \end{cases} \qquad (x \in \mathbb{Z}_p)$$

Then the difference quotients of f are not bounded (for $n \in \mathbb{N}$: $f(p^{2n}) = p^n$, so $|p^{-2n} f(p^{2n})| = p^n \to \infty$ if $n \to \infty$). We show that $f \in M_w(\mathbb{Z}_p)$. Since f is continuous it suffices to show that if x,y,z are $\neq 0$, $|x-y| < |x-z|$ then $|f(x)-f(y)| \leq |f(x)-f(z)|$. This is clear if $|x| = |y|$. If $|x| < |y|$, then $|x| < |y| < |z|$. If $|x| > |y|$, then $|y| < |x| < |z|$. Let $f(x) = a_n$, $f(y) = a_m$, $f(z) = a_t$. Then in both cases $n \neq m$, $t < \min(n,m)$; $|f(x)-f(y)| = |a_n - a_m| \leq |a_t|$ and $|f(x)-f(z)| = |a_n - a_t| = |a_t|$ and we are done.

On the other hand (how surprising is life!)

THEOREM 5.6 *Let k be the field of two elements. Then* $M_w(X) = M_b(X)$.

Proof. We prove that $|x-y| = |y-z|$ implies $|f(x)-f(y)| \le |f(y)-f(z)|$ ($x \neq y$, $y \neq z$, $x,y,z \in X$). There is $\alpha \in K^*$ such that $|\alpha(x-y)| = |\alpha(y-z)| = 1$. So since $k = \mathbb{F}_2$, $\overline{\alpha(x-y)} = \overline{\alpha(y-z)} = 1$, whence $\overline{\alpha(x-z)} = 0$ or $|\alpha(x-z)| < 1$. Thus, $|x-z| < |x-y| = |y-z|$. Since $f \in M_w(X)$, $|f(x)-f(z)| \le \min(|f(x)-f(y)|, |f(y)-f(z)|)$. Consequently, $|f(x)-f(y)| \le \max(|f(x)-f(z)|, |f(z)-f(y)|) \le |f(y)-f(z)|$.

Of particular interest may be monotone functions mapping convex sets onto convex sets.

THEOREM 5.7 *Let K be a local field, let X be a bounded open convex set, and let* $f : X \to X$ *be surjective. Then the following are equivalent.*

(α) $f \in M_b(X)$

(β) $f \in M_s(X)$

(γ) $f \in M_{bs}(X)$

(δ) *f is an isometry.*

Proof. (α) → (β). Since $f(X)$ has no isolated points, f is a homeomorphism, by 5.1(v). Then $f \in M_s(X)$, by 5.3. (β) → (γ). $f^{-1} \in M_b(X)$. We just have shown (α) → (β), so $f^{-1} \in M_s(X)$ i.e., $f \in M_b(X)$.
(γ) → (δ). From the proof of 5.4 we have seen that $|f(x)-f(y)| \le M|x-y|$, where $M = \sup|f(x)-f(y)| = 1$. Hence $|f(x)-f(y)| \le |x-y|$ for all $x,y \in X$, but by the same token this also holds for f^{-1}. Then f is an isometry. (δ) → (α) is obvious.

COROLLARY 5.8 *Let X be an open convex subset of K and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.*

(α) $f \in M_b(X)$

(β) $f \in M_s(X)$

(γ) $f \in M_{bs}(X)$

(δ) f *is a scalar multiple of an isometry.*

Proof. If X is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If X is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (α) (β) (γ) is now easy. To prove (γ) $\to$ (δ) we may assume $f(0) = 0$, $f(1) = 1$. Let $X_n := \{x \in K : |x| \le n\}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is c_n such that $|f(x)-f(y)| = c_n|x-y|$ $(x,y \in X_n)$. By substituting $x = 1$, $y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in M_{ubs}(\mathbb{Z}_p)$ into conditions on the coefficients of f with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(\mathbb{Z}_p)$. So let the notations be as in 3.3(3), and suppose first $f \in M_{ubs}(\mathbb{Z}_p)$ i.e. $|x-y| = |s-t| \iff |f(x)-f(y)| = |f(s)-f(t)|$. Let $n,m \in \mathbb{N}$. If $|n-n_-| = |m-m_-|$ then $|f(n)-f(n_-)| = |f(m)-f(m_-)|$, so if we write $f = \Sigma\lambda_n e_n$ we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0+a_1p+\ldots+a_kp^k$ $(a_k \neq 0)$ then $|n-n_-| = p^{-k}$ where $k = \left[\frac{\log n}{\log p}\right]$. We find

$$\text{if } \left[\frac{\log n}{\log p}\right] < \left[\frac{\log m}{\log p}\right] \text{ then } |\lambda_n| > |\lambda_m|$$

$$\text{if } \left[\frac{\log n}{\log p}\right] = \left[\frac{\log m}{\log p}\right] \text{ then } |\lambda_n| = |\lambda_m|.$$

Moreover, if $\left[\frac{\log n}{\log p}\right] = \left[\frac{\log m}{\log p}\right] = k$ and n-m is ivisible by p^k i.e., $n_- = m_-$ then $|f(n)-f(m)| = |\lambda_n-\lambda_m|$. If n>m then $|f(n-m)-f(0)| = |\lambda_{n-m}| = |\lambda_n|$.

We have found the first half of

THEOREM 5.9 <u>Let</u> $f = \Sigma\lambda_n e_n \in C(\mathbb{Z}_p)$. <u>In order that</u> $f \in M_{ubs}(\mathbb{Z}_p)$ <u>it is necessary and sufficient that condition</u> (*) <u>below holds</u>

$$(*)\quad \begin{cases} |\lambda_n| \text{ is a strictly decreasing function of } \left[\frac{\log n}{\log p}\right] & (n \in \mathbb{N}) \\ \left[\frac{\log n}{\log p}\right] = \left[\frac{\log m}{\log p}\right],\ n \neq m,\ n_- = m_- \text{ implies} & \\ \qquad |\lambda_n-\lambda_m| = |\lambda_n| = |\lambda_m| & (n,m \in \mathbb{N}). \end{cases}$$

We have shown $f \in M_{ubs}(Z_p) \to (*)$. Now suppose (*) and let $|x-y| = p^{-k}$. We show that $|f(x)-f(y)| = |\lambda_{p^k}|$. Now $e_n(x) = e_n(y)$ for $n < p^k$, so

$$f(x)-f(y) = \sum_{n\geq p^k} \lambda_n(e_n(x)-e_n(y)). \text{ Set}$$

$$\begin{aligned} x &:= a_0+a_1p+\ldots+a_kp^k+a_{k+1}p^{k+1}+\ldots \\ y &:= a_0+a_1p+\ldots+b_kp^k+b_{k+1}p^{k+1}+\ldots \end{aligned} \qquad (a_k \neq b_k)$$

Then
$$\left|\sum_{n\geq p^k} \lambda_n e_n(x)\right| = \left|\lambda_{a_kp^k} + \lambda_{a_kp^k+a_{k+1}p^{k+1}} +\ldots\right|$$
$$\left|\sum_{n\geq p^k} \lambda_n e_n(y)\right| = \left|\lambda_{b_kp^k} + \lambda_{b_kp^k+b_{k+1}p^{k+1}} +\ldots\right|$$

Now either a_k or b_k is $\neq 0$, say, $a_k \neq 0$. If $b_k = 0$ then by (*)

$\left|\sum_{n\geq p^k} \lambda_n e_n(y)\right| < \left|\lambda_{a_kp^k}\right| = \left|\sum_{n\geq p^k} \lambda_n e_n(x)\right|$, so $|f(x)-f(y)| = |\lambda_{a_kp^k}| = |\lambda_{p^k}|$. If $b_k \neq 0$ then by (*) $|\lambda_{p^k}| = |\lambda_{a_kp^k}-\lambda_{b_kp^k}| = |f(x)-f(y)|$.

<u>Note</u>. Using similar methods, we can prove: $f = \Sigma\lambda_n e_n$ is in $M_{bs}(\mathbb{Z}_p)$ if and only if we have (**) for all $n,m \in \mathbb{N}$:

$$(**)\qquad \begin{cases} \left[\left[\dfrac{\log n}{\log p}\right] > \left[\dfrac{\log m}{\log p}\right] = k \quad n-m \text{ divisible by } p^k\right] \rightarrow |\lambda_n| < |\lambda_m| \\ \left[\left[\dfrac{\log n}{\log p}\right] = \left[\dfrac{\log m}{\log p}\right] \quad n_- = m_-,\ n \neq m\right] \rightarrow |\lambda_n| = |\lambda_m| = |\lambda_n - \lambda_m|. \end{cases}$$

If we assume only that K has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

THEOREM 5.10 <u>Let X be the unit ball of a discretely valued field. Let $f : X \to X$ be surjective, $f \in M_{bs}(X)$. Then f is an isometry.</u>

<u>Proof</u>. It is clear from previous theory that f is a homeomorphism of the unit ball. It suffices to show that $|f(x)-f(y)| \leq |x-y|$ for all $x,y \in X$. (Apply this result also for f^{-1}. Then f is an isometry.) Let $\pi \in K$, $|\pi| < 1$, be a generator of $|K^*|$. We prove by induction

$$\text{if } |x| = |\pi|^n \text{ then } |f(x)-f(0)| \leq |\pi|^n |f(1)-f(0)|.$$

For $n = 0$ this is clear. ($|x-0| \leq |1-0|$, so $|f(x)-f(0)| \leq |f(1)-f(0)|$). Suppose the statement is true for $n - 1$. Let $|x| = |\pi|^n$. Then $|x-0| < |\pi^{n-1}-0|$, so $|f(x)-f(0)| < |f(\pi^{n-1})-f(0)| \leq |\pi|^{n-1}|f(1)-f(0)|$, so $|f(x)-f(0)| \leq |\pi|^n|f(1)-f(0)|$ and we are done. (In fact, we have shown that a bounded M_s-function has bounded difference quotients.)

Further, from 4.9 we infer

THEOREM 5.11 <u>Let K have discrete valuation and let $f \in M_b(X)$. Then the following conditions are equivalent.</u>

(α) <u>$f(X)$ has no isolated points.</u>

(β) <u>f is injective and continuous.</u>

(γ) <u>f is a homeomorphism $X \overset{\sim}{\to} f(X)$.</u>

Proof. $(\alpha) \to (\gamma)$ is 4.9(ii). $(\gamma) \to (\beta)$ is clear. $(\beta) \to (\gamma)$: if $f(a)$ were an isolated point of $f(X)$, then $\{x : f(x) = f(a)\}$ is open in X. Since f is injective $\{a\}$ is open. But X has no isolated points. Contradiction.

To show that 5.11 may not be true if K has a dense valuation we construct

EXAMPLE 5.12 Let $|K| = [0,\infty)$. Then we construct an M_{bs}-homeomorphism sending

$$\{x \in K : \tfrac{1}{2} < |x| \le 1\} \text{ onto } \{x \in K : 0 < |x| \le 1\}.$$

Proof. Let $\phi : [\tfrac{1}{2},1] \to [0,1]$ be the map $x \mapsto 2(x-\tfrac{1}{2})$ $(x \in (\tfrac{1}{2},1])$. For each $v \in (\tfrac{1}{2},1]$, choose $\beta_v \in K$ such that $|\beta_v| = \frac{\phi(v)}{v}$. Define $f : \{x \in K : \tfrac{1}{2} < |x| \le 1\} \to \{x \in K : 0 < |x| \le 1\}$ as follows

$$f(x) = \beta_{|x|}x \qquad (\tfrac{1}{2} < |x| \le 1)$$

Clearly, $|f(x)| = |\beta_{|x|}|\,|x| = \phi(|x|) \in (0,1]$. The inverse of f is given by $y \mapsto \beta^{-1}_{\phi^{-1}(|y|)}y$, so f is a bijection. Since f^{-1} can be defined in the same way as f (only with ϕ^{-1} instead of ϕ) it suffices to show that $f \in M_s$. Let $|x-y| < |x-z|$.

Suppose $|x| > |z|$. Then $|x-z| = |x|$ and $|y| = \max(|x-y|,|x|) = |x|$. Then $\beta_{|x|} = \beta_{|y|}$, so $|f(x)-f(y)| = \beta_{|x|}|x-y|$ and $|f(x)-f(z)| = |f(x)| = \beta_{|x|}|x| = \beta_{|x|}|x-z|$, so we are done in this case. Suppose $|x| < |z|$. Then $|x-z| = |z|$ and $|y| = \max(|x-y|,|x|) < |z|$. Then $|f(x)-f(y)| \le \max(|f(x)|,|f(y)|) < |f(z)| = |f(z)-f(x)|$.

Suppose $|x| = |z|$. Then $|y| \le \max(|x-y|,|x|) \le |x|$; if $|y|$ were $< |x|$ then $|x-y| = |x| = |z| < |x-z|$, a contradiction, so $|y| = |x| = |z|$, and $|f(x)-f(y)| = \beta_{|x|}|x-y|$, $|f(x)-f(z)| = \beta_{|x|}|x-z|$ whence $|f(x)-f(y)| < |f(x)-f(z)|$.

EXAMPLE 5.13 Extend f to a surjection g of $\{x \in K : |x| \leq 1\}$ onto itself by defining $g(x) = 0$ if $|x| \leq \frac{1}{2}$. We claim that $g \in M_b$. Let $|x-y| \leq |x-z|$. To check wether $|g(x)-g(y)| \leq |g(x)-g(z)|$ we only have to consider the cases $|x| \leq \frac{1}{2}$ and $|y| > \frac{1}{2}$ and $|x| > \frac{1}{2}$ and $|y| \leq \frac{1}{2}$. In the first case, $|x-y| = |y| \leq |x-z|$, so $|z| = \max(|z-x|, |x|) = |z-x| \geq |y|$. Then $|g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)|$. In the second case $|g(x)-g(y)| = |f(x)|$. If $|x| < |z|$ then $|f(x)| < |f(z)| = |f(z)-f(x)| = |g(z)-g(x)|$. If $|x| > |z|$ then $|f(x)| = |g(x)-g(z)|$. If $|x| = |z|$ then $|f(x)-f(z)| = \beta_{|x|}|x-z| \geq \beta_{|x|}|x-y| = \beta_{|x|}|x| = |f(x)|$. Thus we have found <u>a continuous surjection</u> $g : \{x \in K : |x| \leq 1\} \rightarrow \{x \in K : |x| \leq 1\}$, $g \in M_b$, <u>such that</u> $g = 0$ on $\{x : |x| \leq \frac{1}{2}\}$. (Compare 5.11).

<u>EXAMPLE</u> 5.14 <u>Let</u> $h : \{x \in K : |x| \leq 1\} \rightarrow K$ <u>be defined as</u>

$$h(x) := \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \quad (f \text{ as in } 5.12) \\ 0 & \text{if } x = 0. \end{cases}$$

Then h is <u>a non-continuous</u> M_{bs}<u>-function</u>.

<u>Proof</u>. That h is not continuous at 0 is clear. Further, h, restricted $\{x : 0 < |x| \leq 1\}$ is in M_{bs} (see 5.12). Further, since $g \circ h$ is the identity (g as in 5.12), we see that $h \in M_s$. It suffices to show that $|x-y| = |x-z|$ implies $|h(x)-h(y)| = |h(x)-h(z)|$ in case $0 \in \{x,y,z\}$. We may suppose $x \neq y$, $y \neq z$, $x \neq z$. Let $x = 0$. Then $|y| = |z|$, so $|f^{-1}(y)| = |f^{-1}(z)|$ i.e., $|h(x)-h(y)| = |h(x)-h(z)|$. Now let $y = 0$. Then $|x| = |x-z|$. Choose $0 < |t| \leq 1$ such that $|t| < |x|$. Then $|x-t| = |x-z|$ so $|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x)-f^{-1}(z)|$ i.e., $|h(x)| = |h(x)-h(z)|$, and we are done.

6. FUNCTIONS OF BOUNDED VARIATION

In this section X is the unit ball of K, and $B\Delta(X) := \{f : X \to K : \sup_{x \neq y} \left|\frac{f(x)-f(y)}{x-y}\right| < \infty\}$. Let us define

$$\|f\|_\Delta := \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right| : x,y \in X,\ x \neq y\right\} \quad (f \in B\Delta(X)).$$

It will turn out that, in a natural way, $B\Delta(X)$ can be regarded as the space of functions of bounded variation, and that $\| \ \|_\Delta$ plays the role of the total variation.

THEOREM 6.1 Let $f : X \to K$. Then the following are equivalent

(α) $f \in B\Delta(X)$.

(β) f is a linear combination of two increasing functions.

If $|K|$ is discrete (α), (β) are equivalent to

(γ) f is the difference of two bounded monotone functions of some type σ.

(δ) $f \in [M_{bs}(X)]$.

If K is a local field then (α)-(δ) are equivalent to

(ε) $f \in [M_b(X)]$.

(η) $f \in [M_s(X)]$.

Proof. We only prove (α) $\Rightarrow$ (β). The rest follows from (5.10), (5.4). So let $f \in B\Delta(X)$ and choose $\lambda \in K$ such that $|f(x)-f(y)| \le |\lambda||x-y|$ $(x,y \in X,\ x \neq y)$. Then $\lambda^{-1}f$ is a pseudocontraction, $f(x) = \lambda x + \lambda(\lambda^{-1}f(x)-x)$ $(x \in X)$, where $x \to x$ and $x \to \lambda^{-1}f(x)-x$ are increasing.

In the real case, we can define for a function $[0,1] \to \mathbb{R}$, of bounded variation

$$V(f) := \inf\{\mathrm{Var}\, g + \mathrm{Var}\, h : f = g+h,\ g,h \text{ monotone}\}.$$

It is an easy exercise to show that $f \mapsto V(f)$ is a seminorm on the space of all functions of bounded variation and that V is equivalent to the total variation Var, defined via

$$\mathrm{Var}\, f = \sup\{\Sigma|f(x_k)-f(x_{k-1})| : x_0 < x_1 \ldots < x_n \text{ is a partition of } [0,1]\}.$$

So in the non-archimedean situation we define for $f : X \to K$

$$J(f) = \sup\{|f(x)-f(y)| : x,y \in X\}.$$

(If f is considered to be "monotone" then $J(f)$ can be interpreted as the "total variation" of f.) We are led to the following definitions for $f \in B\Delta(X)$:

$$\mathrm{Var}\, f := \inf\{\max(J(g),J(h)) : f = g+h,\ g,h \text{ are scalar multiples of increasing functions}\}.$$

(If $|K|$ is discrete) : $\mathrm{Var}_1 f := \inf\{\max J(g),J(h) : f = g+h,\ g,h$ are in $M_{bs}(X)\}$.

(If K is local) : $\mathrm{Var}_2 f := \inf\{\max J(g),J(h) : f = g+h : g,h \in M_b(X)\}$

$\mathrm{Var}_3 f := \inf\{\max J(g),J(h) : f = g+h : g,h \in M_s(X)\}$.

Let us first compare Var f and $\|f\|_\Delta$. If $f = g+h$ and g,h are scalar multiples of increasing functions we have for $x,y \in X$, $x \neq y$

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le \max\left(\left|\frac{g(x)-g(y)}{x-y}\right|, \left|\frac{h(x)-h(y)}{x-y}\right|\right) \le \max(J(g),J(h))$$

so $\|f\|_\Delta \le \mathrm{Var}\, f$. Conversely, if $|\lambda| > \sup\left|\frac{f(x)-f(y)}{x-y}\right|$ then

$$f(x) = \lambda x + \lambda(\lambda^{-1}f(x)-x) \qquad (x \in X)$$

whence

$$\mathrm{Var}\, f \le |\lambda|$$

So, if $|K|$ is dense we have $\mathrm{Var}\, f = \|f\|_\Delta$ $(f \in B\Delta(X))$. Otherwise we have at least

$$\|f\|_\Delta \le \mathrm{Var}\, f \le c\|f\|_\Delta \qquad (f \in B\Delta(X))$$

(where c is the smallest value > 1).

If $|K|$ is discrete we clearly have $\mathrm{Var}_1 f \le \mathrm{Var}\, f$. Conversely, let $f = g+h$, where $g,h \in M_{bs}(X)$. It follows from the proof of 5.10 that

$$\left.\begin{aligned} |g(x)-g(y)| &\le M|x-y| \\ |h(x)-h(y)| &\le N|x-y| \end{aligned}\right. \qquad (x,y \in X)$$

where $M = \sup|g(x)-g(y)| = J(g)$ and $N = J(h)$.

So

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le \max(J(g),J(h)), \text{ whence}$$

$$\|f\|_\Delta \le \mathrm{Var}_1 f.$$

Similar proofs work for $\mathrm{Var}_2 f$, $\mathrm{Var}_3 f$. We have

THEOREM 6.2 <u>The seminorms</u> Var, Var_1, Var_2, Var_3, <u>on</u> $B\Delta(X)$ <u>(whenever defined) are all equivalent to</u> $\|\ \|_\Delta$.

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